Ether Squirts.

Being an Attempt to specialize the form of Ether Motion which forms an Atom in a Theory propounded in former Papers.

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RÉSUMÉ.

In three previous papers (Camb. Phil. Trans., Vol. XIV, page 71; London Math. Society's Proceedings, Vol. XX, p. 38 and p. 297) I have developed the results which flow from supposing the ultimate atom to be a sphere pulsating in a perfect fluid. I have shown that this hypothesis is not without suggestion for the phenomena of chemical affinity, cohesion, and spectrum analysis in the first paper; that it can be applied to explain dispersion and other optical phenomena, as well as certain electrical and magnetic phenomena in the second paper; while the fact that it leads to generalized elastic equations is developed in the third paper. In the present memoir I have endeavored to show that all these results still hold good if the pulsating sphere be replaced by an ether squirt which resists variations in its rate of flow. From whence the ether flows and why its flow resists variations are problems which, as they fall outside the range of physics, I leave to the metaphysicians to settle. The ether squirt as a model dynamic system for the atom seems at any rate to possess the property of simplicity. But the action of one group of ether squirts upon a second group leads to equations the complexity of which seems quite capable of paralleling any intricacy of actual Nature. The main portion of this paper is devoted to an investigation of inter-atomic and inter-molecular forces, and brings out the striking influence in producing cohesion of 'kin-atoms' in different molecules. Thus if A_2B and $A_2'B'$ represent two molecules compounded of two A atoms and one B atom, it is shown that the inter-atomic action of A on A in the first molecule is largely influenced by the action of A' in an adjacent molecule. In many cases the term which A''s action

produces in the atomic action of A on A is as important as that of A upon B. The effect of this action of kin-atoms in different molecules is shown to lead to the 'hypothesis of modified action' and to 'multi-constant' equations of elasticity. Its denial of the *literal* truth of the Second Law of Motion is also discussed, and its application to certain general problems of cohesion dealt with.

The law of gravitation and the theory of the potential are shown to be more intelligible on the ether squirt theory than on that of the pulsating sphere as developed in §§51–9 of my first paper.

ON A CERTAIN ATOMIC HYPOTHESIS—ETHER SQUIRTS.

1). "It has become the fashion," said Professor Fitzgerald in his Bath address to the British Association, "to indulge in quaint cosmical theories and to dilate upon them before learned societies and in learned journals. I would suggest, as one who has been bogged in this quagmire, that a successor in this chair might well devote himself to a review of the cosmical theories propounded within the last few years. The opportunities for piquant criticism would be splendid."

The pleasure of "bogging oneself in this quagmire" is so great that even piquant criticism cannot restrain me from adding another quaint cosmical theory to the many that already exist. This hypothesis may be briefly summed up in the statement: *that an atom or the ultimate element of ponderable matter is an ether squirt.* (See Clifford's use of the term 'squirt,' *Elements of Dynamic*, I, p. 214.)

2). As it is well known, the vortex theory of matter reduces the ether to a perfect fluid, and endeavors to build up matter by some form of motion in this fluid. The infinite variety of motions which a perfect fluid is capable of, suggest all sorts of rotational or even irrotational forms which may account for matter. The great beauty of all such fluid motion solutions is their reduction of the physical universe to a single imponderable medium; they avoid dualistic explanations of natural phenomena. As, however, no sufficiently simple vortex origin of matter has yet been formulated, many scientists, notably Sir William Thomson, have, for the practical purpose of explaining optical phenomena, fallen back for the present on mechanical molecules. The mechanical molecule of Thomson, while of great assistance to the understanding of many facts of dispersion and absorption, undoubtedly possesses chemical disadvantages; it presents no obvious mode of disassociation. On the other hand, Lindemann has recently shown that it can

be used effectively to explain various phenomena of magnetism and electricity. He, however, assumes the ethereal medium to be of the nature of a *perfect fluid.* The Thomson-Lindemann atoms and molecules thus show us so far only complex mechanisms, and raise the not unnatural repugnance of the philosophical mind to a dualistic theory of the universe.

3). In a paper published in the Camb. Phil. Trans. (Vol. XIV, page 71), I have endeavored to explain certain optical and chemical phenomena by treating the ether as a perfect fluid and supposing the atom to be a differentiated part of the ether, which, as a first approximation, may be considered of spherical form and the surface of which is capable of pulsation. I did not venture upon any suggestion as to the nature of this differentiated portion of the ether. I merely suggested that it possibly might be .explained as an ether vacuum with surface energy. All that I required was a surface in the ether approximately spherical and capable of pulsation. In a later paper I have shown that most of the results which flow from the Thomson-Lindemann atom, together with a ready explanation of disassociation, could be obtained from the pulsating atom. The molecule based upon it seemed to me to have distinct advantages over the complex mechanism of their spring-shell molecule. In the present paper I suggest a kinematic fluid origin for the pulsating spherical atom. The sort of fluid motion which I have chosen is an irrotational one, and therefore one capable of far more ready handling than a vortex-sponge system. The difficulty of course arises as to how rays of light are to be propagated by transverse vibration in such a fluid medium, but then that difficulty also occurs in the vortex theory when it treats the ether outside the vortex atom or matter as a perfect fluid. It may also be urged against Lindemann's arguments which largely depend on the perfect fluid motion of the medium outside the spring-shell atom. Perhaps the perfect fluid is turbulent.

At the same time it may be remarked that our theory does not absolutely require the ether to be a perfect fluid. It supposes only: (1) that its velocity at any point is the differential of a certain function ψ with regard to the direction of the velocity, or if u, v, w be the velocity components,

$$u = \frac{d\psi}{dx}, \quad v = \frac{d\psi}{dy}, \quad w = \frac{d\psi}{dz}, \tag{1}$$

(2) that the ether is incompressible and continuous except at points occupied by 'matter,' or

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{2}$$

Lastly, it assumes: (3) that the kinetic energy of the *whole system* is given by the same expression as in the case of a perfect fluid; that is,

$$\text{Kinetic Energy} = -\tfrac{1}{2}\rho\int\psi\frac{d\psi}{dn}\,d\sigma, \tag{3}$$

where the integration is to be over the surface of all 'matter,' and n is the direction of the normal at the element $d\sigma$ of such surface, its positive sense being toward the fluid, ρ being the density of the medium. Whatever the nature of the ether, this is obviously an irrotational motion, but beyond the above assumptions it is not necessary to consider the ether as identical with a perfect fluid.

4). Now the results of the paper to which I have referred above were obtained by supposing the spherical atom to have certain surface pulsations; that is, the velocity of the fluid at the surface of the atom was supposed to be given and equal to the surface velocity of the particle of 'matter' there situated. Now we should have obtained precisely the same equations had we supposed a flow of the ether over the surface of the atom with the same velocity as that with which we endowed the atomic surface. Now this might arise from two causes, either (a) we may suppose a source or sink of ether inside the atom, the ether itself being incompressible; or (b) we may suppose the ether at the atom to be compressible, and the atom might be treated as an element of ether in the 'gaseous state,' a bubble of ether vapor in ether—just as we might have a bubble of the vapor of a substance in its liquid at certain temperatures and pressures. The difficulty of this latter explanation is really its dualistic character; it throws back the explanation of gas and liquid on something pre-ethereal and tends to suggest a second ether to explain the first.

On the other hand, our first hypothesis, especially when we simplify our atom by reducing it to a point, so that it really coincides with the source or sink, has a good deal to recommend it; it very much reduces the terms which appear in the expression for the kinetic energy of both atoms and ether; it retains, however, the terms that appear to be really necessary, and it renders these terms absolutely accurate instead of only approximations. It is, supposing it will suffice to explain physical phenomena, an extremely simple monistic hypothesis: the fluid medium in irrotational motion is the primary substance, the atom or element of matter is a squirt of the same substance. From whence the squirt comes into three-dimensioned space it is impossible to say; the theory limits our possibility of knowledge of the physical universe to the existence of the squirt. It

may be an argument for the existence of a space of higher dimensions than our own, but of that we can know nothing, and we are only concerned with the flow into our own medium, with the ether squirt which we propose to term 'matter.' Some idea of the theory may be obtained in the following manner: Suppose a perfectly smooth metal plate on which are placed any number of electrodes capable of free motion on the surface of the plate, then if these were sources and sinks for electricity passing in and out of the plate, they would move each other about, and in doing so they would move the really immaterial sinks and sources in the plate. To a being in the flatland provided by the electric medium of the plate, these really immaterial sinks and sources would appear to be centres which mutually accelerated each other, i. e. he would suppose them endowed with force, or, as seats of force, he would properly term them matter. The ether squirt in three-dimensioned space is endowed in the same way with the properties of 'matter.' It possesses, besides, all the singular merit of the Boscovichian atom, i. e. it is for theoretical purposes a mathematical point.

Of course we cannot understand what is the exact meaning of both the ether potential ψ and the ether velocity becoming infinite at the squirt or element of matter, but this difficulty would be a like puzzle to the dweller in the electric plate to which I have previously referred.

If the ether be incompressible, the total flow in and out at all squirts must be zero, or if this flow be proportional to the 'mass,' we find the total quantity of matter in the universe is constant, namely, *zero*. This would involve the existence of negative as well as positive matter in the universe, but the law of gravitation as deduced from our theory tells us that there must be a large preponderance of matter of like sign in that part of the universe with which we are acquainted. It is quite possible, however, that matter outside the solar system may be of a different sense, or even perhaps the ring of a planet be composed of matter of a different sense to the matter of the planet itself. The principle of the indestructibility of matter would reduce to the obvious theorem that we cannot stop the flow of any squirt, without a like flow coming into our space somewhere else—experience tells us, in the immediate neighborhood of the old squirt. We can vary the flow of any squirt and so obtain positive and negative electricity, but we cannot, so far as our present experience goes, create positive and negative matter—an ether source and a sink—at a point where there was previously no squirt at all.

5). In the following discussion the flow of any squirt will not be considered a constant, only its mean value over some long or short interval will be; thus the expression for the flow may contain periodic terms due either to variations characteristic of the nature of the particular squirt, or to variations forced upon it by the vibratory character of the flow of adjacent squirts. Thus if V_s be the velocity of flow of the ether just by the sth squirt, we shall have

$$V_s = {}_0v_s + {}_1v_s \sin(n_1 t + \alpha_1) + {}_2v_s \sin(n_2 t + \alpha_2) + {}_3v_s \sin(n_3 t + \alpha_3) + \text{etc.} \quad (4)$$

Here the constant term will be used to explain gravitation, the enforced periodic variations to throw light on chemical affinity and cohesion, while other vibrational terms will be suggestive for optical and electrical phenomena.

6). Let $r_1, r_2, r_3, \ldots . r_s, \ldots .$ be the distances of any point in the ether from the 1st, 2d, 3d, ... s^{th}, ... squirts, then a suitable solution of equations (1) and (2) is

$$\psi = \frac{A_1}{r_1} + \frac{A_2}{r_2} + \frac{A_3}{r_3} + \ldots . + \frac{A_s}{r_s} + \ldots . , \quad (5)$$

where $A_1, A_2, A_3, \ldots . A_s, \ldots .$ are any constants. To determine these, let $Q_1, Q_2, \ldots . Q_3, \ldots .$ be the instantaneous rates at which ether is flowing in at the 1st, 2d, ... s^{th}, ... squirts, then we have

$$Q_s = \int \frac{d\psi}{dr_s} r_s^2 d\omega = 4\pi r_s^2 V_s \rho,$$

where r_s is to be taken very small and the integration is to include all the solid angle round the sth squirt. Since r_s^2 is to be taken very small, we find for the limit

$$Q_s = -4\pi A_s,$$

or

$$\psi = -\frac{1}{4\pi\rho} \left\{ \frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3} + \ldots . + \frac{Q_s}{r_s} + \ldots . \right\}. \quad (6)$$

Now if Q_s were finite, V_s would become infinite when we make r_s vanishingly small. In order to avoid this, we shall assume V_s to be the radial velocity of the flow on the surface of a very small sphere of radius a_s surrounding the squirt, this sphere being taken so close that it only encloses its own squirt, and that the whole flow may be considered radial and uniform over its surface. We thus find

$$V_s = Q_s / 4\pi a_s^2 \rho . \quad (7$$

We proceed now to determine from (3) the entire kinetic energy of the system; we have, $d\omega$ being an element of solid angle,

$$K.E. = \frac{1}{8\pi}\Sigma\int\left(\frac{Q_1}{r_1}+\frac{Q_2}{r_2}+\ldots\ldots+\frac{Q_s}{r_s}+\ldots\ldots\right)$$
$$\times\frac{1}{4\pi\rho}\left(\frac{Q_1}{r_1^2}\frac{dr_1}{dr_s}+\frac{Q_2}{r_2^2}\frac{dr_2}{dr_s}+\ldots\ldots+\frac{Q_s}{r_s^2}+\ldots\ldots\right)r_s^2\,d\omega,$$

the summation extending over all values of s from 1 to ∞ and r_s being put in each case very small in the limit. Thus we have

$$K.E. = \frac{1}{32\pi^2\rho}\Sigma Q_s\int\left(\frac{Q_1}{r_{1s}}+\frac{Q_2}{r_{2s}}+\ldots\ldots+\frac{Q_s}{r_s}+\ldots\ldots\right)d\omega$$
$$=\frac{1}{8\pi\rho}\Sigma Q_s\left(\frac{Q_1}{r_{1s}}+\frac{Q_2}{r_{2s}}+\ldots+\frac{Q_{s-1}}{r_{s-1,s}}+\frac{Q_{s+1}}{r_{s+1,s}}+\ldots\right)+\frac{1}{32\pi^2}\Sigma\int\frac{Q_s^2}{r_s}\,d\omega$$
$$=\frac{1}{4\pi\rho}\Sigma\frac{Q_sQ_{s'}}{\gamma_{ss'}}+\frac{1}{8\pi\rho}\Sigma\frac{Q_s^2}{a_s},$$

where in the first summation s and s' are to take all possible different values and $\gamma_{ss'}$ is the distance between the sth and s'^{th} squirts, and in the second summation we are to sum for all squirts. The terms of the last summation would become infinite if we did not exclude the volume of the fluid absolutely at the squirt. Substituting for Q_s in terms of V_s, we find

$$K.E. = \Sigma\, 2\pi\rho a_s^3 V_s^2 + \Sigma\frac{4\pi\rho a_s^2 a_{s'}^2}{\gamma_{ss'}}V_s V_{s'}. \tag{8}$$

This is the total energy of the ethereal medium excluding small spherical elements in the immediate neighborhood of the squirts. It agrees with the result obtained in Art. 14, p. 82, of the first paper on *pulsating* atoms (Camb. Phil. Trans., Vol. XIV) as a *first approximation* to the kinetic energy. It has therefore two advantages over the result given in that paper: it is no longer an approximation, but accurate so far as the hypothesis reaches; and farther, V_s need not be, like the $\dot{\phi}_0$ of the notation of that paper, merely periodic; it may contain a constant term, which would have involved rupture in the case of the pulsating spherical atom.

7). Let us deal first with the mean velocity of flow from the squirt across the surface of the small sphere of radius a_s. This we have represented by ${}_0v_s$. Since this does not vary with the time, it cannot arise in any equations—such as the Hamiltonian equations—deduced from the kinetic energy by varying the time. Hence it will not arise when we consider the forced variations in the flow

of the squirts. It contributes terms to the kinetic energy of the system depending upon the mutual distances of the squirts

$$= \Sigma \frac{4\pi\rho a_s^2 a_{s'}^2 \, {}_0v_s \, {}_0v_{s'}}{\gamma_{ss'}}$$

$$= \frac{1}{4\pi\rho} \Sigma \frac{q_s q_{s'}}{\gamma_{ss'}},$$

if q_s be the mean rate at which ether is poured in at the sth squirt.

Now let us define 'mass' to be the "mean rate at which ether is poured in at any squirt." Then we have the following result:

There is an attractive force between any one element of matter and any other element of matter, varying directly as their masses and inversely as the square of their distance.

This is the ordinary law of gravitation. The constant $1/4\pi\rho$ may either be taken as a constant of attraction or in the form $\sqrt{1/4\pi\rho}$ as a factor of mass.

If we write $\sqrt{1/4\pi\rho}\, q_s = m_s$, we have, confining our attention to one element of matter m_s, the following terms in the kinetic energy:

$$m_s \Sigma \frac{m_{s'}}{\gamma_{ss'}}.$$

Now this element of the kinetic energy of the ether appears as potential energy when we leave the ether out of sight—thus, the potential energy of gravitating bodies is disguised kinetic energy of the ether. The potential of a number of elementary masses at any point not occupied by one of them is of the form

$$\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} + \ldots\ldots,$$

or remembering the values of m_1, m_2, etc., we see that this is equal to the *mean* value of $-\psi\sqrt{4\pi\rho}$ for the same set of elementary masses: see equation (6). Thus the gravitational potential differs only by a change in sign and a constant factor from the velocity potential of the ether at the same point. It thus obtains a definite physical meaning with regard to the ether. Further, we note that if an element of matter (or a squirt) be placed at any point of a field in which there are other gravitating elements, their effect on the new element is to cause it to move along the line of flow of the ether at that point due to the gravitating system; that it moves with an acceleration proportional to the speed of the ether flow but in an *opposite* sense. Most of the properties of the potential are thus capable

of being stated in language involving either the kinetic energy or local motion of the ether.

8). Leaving the gravitational terms, let us endeavor to form equations for the forced vibrations in a set of ether squirts. Here a difficulty arises at the very outset, owing to our ignorance of why and how the ether is injected into the ethereal medium. Let us suppose p_s to represent the amount of ether injected from a given epoch up to the instant in question through the sth squirt, then obviously the time variation of p_s, or $\dot{p}_s$, is what we have written Q_s. Thus the kinetic energy may be written in the form

$$K.E. = \frac{1}{8\pi\rho}\Sigma\frac{\dot{p}_s^2}{a_s} + \frac{1}{4\pi\rho}\Sigma\frac{\dot{p}_s\dot{p}_{s'}}{\gamma_{ss'}}. \tag{9}$$

Now the equations for the forced vibrations in the flow of the squirt will be meaningless unless the squirt tends to *resist* variations in its rate of flow. But we do not know why it flows, much less the reason why only a limited variation is permissible in its rate of flow. That depends on the state of affairs outside the space which is alone sensible to us and with which we can deal. We can, as it were, only guess at the 'potential energy' of our atom, which lies outside our space. On the ether squirt hypothesis, the mechanism of the *Ding-an-sich* is beyond our control or inspection. One thing, however, is obvious, if the theory is to apply to facts as we understand them, only a slight change in the rate of flow is possible, and the phenomena of the spectrum, especially on the "single bright line" theory of an absolutely disassociated atom, lead us to believe that this variation is periodic and characteristic for each individual atom. This suggests and in part justifies our adding to the total energy of the system a potential energy of the form

$$\tfrac{1}{2}\Sigma\tau_s'(p_s - q_s t)^2.$$

For this gives a 'force' at the sth squirt tending to *resist* variation of its rate of flow equal to

$$\tau_s'(p_s - q_s t),$$

or proportional to the amount which the forced vibration has drawn from the ether-store of the particular squirt over and above its ordinary gravitation allowance. We have termed this 'potential energy,' but just as we have banished potential energy by our hypothesis from the space under our control, so we might replace this by kinetic energy if we were able to include the unknown system outside our space.

Thus the difference L of the kinetic and potential energies of the system, supposing *there to be no motion of translation*, is given by

$$L = \frac{1}{8\pi\rho}\Sigma\frac{\dot{p}_s^2}{a_s} + \frac{1}{4\pi\rho}\Sigma\frac{\dot{p}_s\dot{p}_{s'}}{\gamma_{ss'}} - \tfrac{1}{2}\Sigma\tau'_s(p_s - q_s t)^2.$$

Hence, applying Lagrange's equations, we obtain for the typical vibrational equation

$$\frac{1}{4\pi\rho a_s}\ddot{p}_s + \frac{1}{4\pi\rho}\Sigma\frac{\ddot{p}_{s'}}{\gamma_{ss'}} + \tau'_s(p_s - q_s t) = 0.$$

Let $\left.\begin{array}{l}\phi_s = p - q_s t \\ \tau_s = 4\pi\rho\tau'_s\end{array}\right\}$, then we have

$$\ddot{\phi}_s/a_s + \Sigma\ddot{\phi}_{s'}/\gamma_{ss'} + \tau_s\phi_s = 0. \tag{10}$$

This is practically identical with the typical equation by means of which I have endeavored, in Arts. 14–25 (pp. 82–92) of my first paper, to represent the main facts of spectrum analysis. It is also the equation which leads on a fairly plausible hypothesis to a formula for the dispersion of light passing through a medium composed of such ether squirts exactly similar to that found by Thomson and Lindemann in the case of a spring-shell molecule (see my second paper, London Math. Soc. Proc., Vol. XX, p. 41). So far, then, the ether squirt is equally efficient with the pulsating spherical atom (or, for the matter of that, with the spring-shell molecule) in explaining optical phenomena. It is obviously much more suggestive for the problem of gravitation. We must turn now to its bearing on atomic and cohesive forces.

9). *To find the atomic force between the sth and s'^{th} atoms of a group of k atoms forming a molecule.*

Looking at equation (10), we observe that if two atoms were at a distance which was great compared with the radii of the small spheres by which we have enclosed each squirt (we may take these radii equal if we please; they are not atomic radii,' for our atom is really a Boscovichian point), then the summation term would vanish as compared with the other, and the flow of the squirt vary with its own characteristic period $2\pi/\sqrt{\tau_s a_s}$, or $2\pi/\nu_s$ say. Hence in this case

$$p_s = q_s t + C_s\cos(\nu_s t + \alpha_s).$$

We have now to solve k equations of the type (10) which may be written

$$\ddot{\phi}_s + \Sigma\ddot{\phi}_{s'}\frac{a_s}{\gamma_{ss'}} + \nu_s^2\phi_s = 0. \tag{11}$$

Assume $\phi_s = B_s \cos(nt + \beta)$, then

$$B_s(\nu_s^2 - n^2) - \Sigma B_{s'} n^2 \frac{a_s}{\gamma_{ss'}} = 0,$$

or

$$B_s\left(\frac{1}{\nu_s^2} - \frac{1}{n^2}\right) + \frac{1}{\nu_s^2} \Sigma B_{s'} \frac{a_s}{\gamma_{ss'}} = 0. \qquad (12)$$

This gives to determine $\frac{1}{n^2}$ the determinantal equation

$$\begin{vmatrix} 1 - \frac{\nu_1^2}{n^2}, & \frac{a_2}{\gamma_{12}}, & \frac{a_3}{\gamma_{13}}, \ldots & \frac{a_k}{\gamma_{1k}} \\ \frac{a_1}{\gamma_{12}}, & 1 - \frac{\nu_2^2}{n^2}, & \frac{a_3}{\gamma_{23}}, \ldots & \frac{a_k}{\gamma_{2k}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_1}{\gamma_{1k}}, & \frac{a_2}{\gamma_{2k}}, & \ldots\ldots & 1 - \frac{\nu_k^2}{n^2} \end{vmatrix} = 0. \qquad (13)$$

The roots of n obtained from this equation substituted in (12) give the constants B_s. In order to obtain some idea of the nature of the periods and the constants we may find approximations to their value in series ascending by powers of a/γ. We easily find as a first approximation to the sth root of the determinant

$$\frac{1}{n_s^2} = \frac{1}{\nu_s^2} - \Sigma \frac{a_s a_{s'}}{\gamma_{ss'}^2} \frac{1}{\nu_s^2 - \nu_{s'}^2}, \qquad (14)$$

the summation extending to all values of s' other than s. Further, if B_s represent the amplitude of the sth periodic vibration in the sth atom, we find

$$B_{s'} = -\frac{B_s \nu_s^2}{\nu_s^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{ss'}}. \qquad (15)$$

Thus we have

$$\phi_{s'} = B_{s'} \cos(n_{s'} t + \beta_{s'}) - \Sigma \frac{B_s \nu_s^2}{\nu_s^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{ss'}} \cos(n_s t + \beta_s), \qquad (16)$$

the summation extending to all values of s other than s'. Thus finally we have

$$\dot{p}_{s'} = q_{s'} - n_{s'} B_{s'} \sin(n_{s'} t + \beta_{s'}) + \overset{(s \text{ all values but } s')}{\Sigma} \frac{n_s B_s \nu_s^2}{\nu_s^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{ss'}} \sin(n_s t + \beta_s),$$

$$\dot{p}_s = q - n_s B_s \sin(n_s t + \beta_s) + \overset{(s' \text{ all values but } s)}{\Sigma} \frac{n_{s'} B_{s'} \nu_{s'}^2}{\nu_{s'}^2 - \nu_s^2} \frac{a_s}{\gamma_{ss'}} \sin(n_{s'} t + \beta_{s'}).$$

We must now return to equation (9) and substitute these values of $\dot{p}_s$ and $\dot{p}_{s'}$ in the term which involves $1/\gamma_{ss'}$. This will give us the *apparent* force function U between the atoms. In order to ascertain its measurable value we must take its mean with regard to an interval of time embracing many vibrations, we easily find

$$\text{mean of } \dot{p}_{s'}\dot{p}_s = q_{s'}q_s - \tfrac{1}{2}\frac{n_s^2\nu_s^2 B_s^2}{\nu_s^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{ss'}} - \tfrac{1}{2}\frac{n_{s'}^2\nu_{s'}^2 B_{s'}^2}{\nu_{s'}^2-\nu_s^2}\frac{a_s}{\gamma_{ss'}} + \tfrac{1}{2}\sum_{(r \text{ all values but } s \text{ or } s')}\frac{n_r^2 B_r^2\nu_r^4}{(\nu_r^2-\nu_{s'}^2)(\nu_r^2-\nu_s^2)}\frac{a_s a_{s'}}{\gamma_{rs'}\gamma_{rs}}.$$

The first term here, $q_{s'}q_s$, we may omit; it gives rise only to the gravitation potential with which we have already dealt. Retaining terms only as far as the cube of a/γ, we have

$$\begin{aligned}\text{mean of } \dot{p}_s\dot{p}_{s'} &= \tfrac{1}{2}\frac{a_{s'}\nu_s^4B_s^2 - a_{s'}\nu_{s'}^4B_{s'}^2}{\nu_{s'}^2-\nu_s^2}\frac{1}{\gamma_{ss'}}\\ &+ \tfrac{1}{2}\frac{a_{s'}\nu_s^6B_s^2 + a_s\nu_{s'}^6B_{s'}^2}{(\nu_s^2-\nu_{s'}^2)^2}\frac{a_s a_{s'}}{\gamma_{ss'}^3}\\ &+ \tfrac{1}{2}\sum_{(r \text{ all values but } s \text{ or } s')}\frac{\nu_r^6B_r^2}{(\nu_r^2-\nu_{s'}^2)(\nu_r^2-\nu_s^2)}\frac{a_s a_{s'}}{\gamma_{rs'}\gamma_{rs}}.\end{aligned}$$

Thus for the *attractive* atomic force between the sth and s'^{th} atom we have, from $F_{ss'} = -dU/d\gamma_{ss}$,

$$\left.\begin{aligned}F_{ss'} &= \frac{a_{s'}\nu_s^4B_s^2 - a_s\nu_{s'}^4B_{s'}^2}{4\pi\rho\,(\nu_{s'}^2-\nu_s^2)}\frac{1}{\gamma_{ss'}^3}\\ &+ \frac{a_{s'}\nu_s^6B_s^2 + a_s\nu_{s'}^6B_{s'}^2}{2\pi\rho\,(\nu_s^2-\nu_{s'}^2)^2}\frac{a_s a_{s'}}{\gamma_{ss'}^5}\\ &+ \frac{1}{8\pi\rho}\sum\frac{\nu_r^6B_r^2}{(\nu_r^2-\nu_{s'}^2)(\nu_r^2-\nu_s^2)}\frac{a_s a_{s'}}{\gamma_{ss'}^2\gamma_{rs}\gamma_{rs'}}\end{aligned}\right\}. \qquad (17)$$

This is the value of the inter-atomic force so far as the fifth power of the inverse of the atomic distance. Thus we find—

(i). Atomic force varies partially as the inverse cube, partially as the inverse fifth power of the distance.

(ii). The modifying action of other atoms of the same molecule (or of atoms within the range of chemical action) varies as the inverse *fourth* power of atomic distance, but only as the inverse *square* of the particular atomic distance of the two atoms between which we are measuring the force.

These results are in complete agreement with those obtained for *Atomic Forces* in Arts. 31–41 (pp. 96–104) of my first paper. The 'chemical intensity' and the 'chemical affinity' may be defined in precisely the same manner as in Art. 38 and the same explanations given of the laws of chemical combination and disassociation. Squirts act for atomic forces exactly as the pulsating spheres dealt with in that paper.

10). Some general remarks may, however, be made on the formula (17) for inter-atomic force. I have in this paper proceeded to a higher approximation than in Art. 37 of the first paper, and we see it introduces a term into the force between the sth and s'^{th} atom which is next in importance to the cubic term and depends upon the distances of those atoms from the rth atom. Thus what Jellett has termed the "hypothesis of modified action" holds for a system of ether squirt atoms. Now this warns us of the care which must be used in applying the literal interpretations of the definition of Force and of the Second Law of Motion to atoms. Let us examine this somewhat more closely.

Force is any cause which tends to alter a body's natural state of rest, or of uniform motion in a straight line.

Now the s'^{th} atom is a cause of such change of motion in the sth atom. It exerts a 'force' upon it depending upon their relative positions.

Now the *Second Law* of motion is often stated in the following manner:

Whenever any forces whatever act on a body, then, whether the body be originally at rest or moving with any velocity and in any direction, each force produces in the body the exact change of motion which it would have produced if it had acted singly on the body originally at rest.

It is the last phrase of this law which gives rise to doubt. Force being defined as the cause of motion, it might be supposed from the law that we could superpose causes of motion without altering the effects they produce on the motion of the body when they act alone. But this is not true when we deal with atoms within the range of chemical action. The introduction of a third atom into a field containing two others not only introduces new forces between those two atoms, but profoundly modifies the force existing between the two first atoms. This was an impossibility so long as force was considered as something *inherent in matter* and the Second Law of Motion was intelligible in its literal sense, but the moment we throw back force on the kinetic energy of the ethereal medium, then the force between any two elements is dependent not

solely on the atoms but on the whole ethereal field. The true way out of the difficulty is to disregard the second law and treat the instantaneous acceleration of any element as a function of its position with regard to the whole of the surrounding field. Nor is this alone sufficient; the force cannot in another sense be identified with the cause of change in motion. The cause of acceleration of a particular atom in the ether depends not only on its position in the field, but also *on its own motion*, on its own periodic translatory and internal vibratory motions as well as on the similar motions of other atoms. Hence, if force be identified with the 'cause of change in motion,' it does not produce the same change when we place the atom in a state of complete rest. In fact, if we once got an atom into a state of complete rest, it would denote complete rest of the ether in its immediate neighborhood, and thus no force could or would be 'acting upon it.' In fact there is little doubt that the atom would have ceased to be when thus brought to rest. There seems to me therefore considerable danger in the literal application of Newton's definition of force and his laws of motion to the mutual acceleration of atoms. They seem to exclude the 'hypothesis of modified action' as well as the existence of a generalized strain-energy, both of which I hope to show in a future paper arise from inter-molecular force involving the speed of the molecules; such terms in inter-molecular force are really due to the kinetic energy of the ether. (See my third paper, London Math. Soc. Proceedings, Vol. XX, p. 297.)

11). Returning to formula (17), I need not rediscuss the possibilities it offers for explaining chemical actions, but I will draw attention to a new range of phenomena which possibly the introduction of the modifying term will throw some light upon. We will not consider the action of a third or fourth atom brought close to a molecule in breaking it up as in the first paper, but we will start with a molecule of k atoms and ask what physical changes can disassociate it. The force as the inverse fifth of the inter-atomic distance is always attractive, but it may be considered small as compared with the inverse cube term. On the other hand, a single term of the modifying force, while it varies as the inverse square of the distance, is yet of the inverse fourth order in mean inter-atomic distance. There are, however, any number of such terms, so that it may rise to equal importance with the cubic term if we take a sufficient number of atoms. Further, it has for its numerator a simple square of an amplitude, while the cubic term has the *difference* of two quantities involving the amplitudes squared. On both of these counts, then, they cannot in many cases be so widely different in

magnitude, and slight variations in their relative magnitude may assist us in explaining various phenomena. I take two theoretical examples.

(i). A and B are two atoms whose 'chemical affinity' (see Art. 38 of my first paper) is negative. They cannot enter into a chemical combination. This means that the coefficient of the cubic term is negative. If their 'antipathy' for each other be not very great, a third atom C (the 'r^{th} atom') may bring them into chemical union. For this purpose it is necessary that the modifying term should be positive and greater than the cubic term. In order that it should be positive, however, we must have ν_r either greater or less than *both* ν_s and $\nu_{s'}$, or the single bright line of the atom which would link A and B together must not lie between the characteristic bright lines of A and B. On the other hand, if the 'antipathy' of A to B is somewhat greater we may require two, three or more atoms to hold them together by their modifying action on A and B's inter-atomic force. Thus ABC might be an impossible chemical union, but ABC_2 or ABC_3 possible. Further, since the amplitudes B_s, $B_{s'}$, B_r, etc., are probably functions of the surrounding physical conditions, as pressure, temperature, etc., it follows that under certain physical conditions ABC_3 might be possible and ABC_2 not, etc., etc.

The disassociating effect of a third atom C on A and B when the chemical affinity of the latter is positive, may be discussed in like manner. In this case C's bright line must lie between those of A and B. Of course A and B, although they may now repel each other, may be still held together by the affinity of both for C, but this chemical combination will have very different physical properties from one in which A and B have themselves affinity.

(ii). Suppose the atoms A, B, C in chemical union, then if the modifying term be negative and the cubic term (between A and B) be positive, it is conceivable that a change in the physical condition—especially a change in the temperature—without the introduction of any further atoms, will invert the order of magnitude or produce an antipathy between A and B. We may thus state the principle: If three atoms, A, B, C, are in chemical union, and A have a real chemical affinity for B, this affinity will be strengthened by the presence of the atom C if its bright line does not lie between the characteristic bright lines of A and B, but the affinity on the other hand will be weakened if the line does lie between those of A and B. Anything that tends to excite C or to cause its amplitude to vary more rapidly than those of A and

B, e. g. a wave of light or heat nearly of the period of its characteristic bright line—will thus in the corresponding cases largely strengthen or weaken the affinity of A for B. It may produce a more stable compound or it may lead to disassociation. Thus the effect of heat on a chemical compound receives considerable light from the existence of the modified action term.

12). We now pass to molecular force, or to the force between groups of atoms constituting two separate molecules. The peculiarity of such groups is that the flow variation of an atom of one group is practically only affected by the atoms of its own group. If the atoms of one group are essentially affected by the atoms of another, then the groups really enter into a sort of partial chemical combination with each other. This, which is hardly a possibility in the case of a rare gas, might, on the other hand, be to some extent true for a solid or liquid body. The cohesive forces might be to some extent chemical forces. Possibly in certain loose compound substances molecules may be chemically compounded, but the drift of experiment seems to be against this sort of union as a rule; inter-molecular distance seems to be considerable as compared with inter-atomic distance, and the force between molecules to be thus of a different nature to that between atoms.* At the same time the phenomena of fluted and continuous spectra warn us that it is quite possible that for a full explanation of even inter-molecular action we may be thrown back on something of the nature of a chemical action between molecules.

Let us suppose we have three molecules of the same substance each containing k atoms, and let r, r' and r'' denote any triplet of corresponding atoms of the three. We have first to determine how far the vibrations of any one atom are influenced by the existence of the other two groups of atoms. To do this we have to find the $3k$ periods given by the characteristic determinant like (13), but in this special case every three roots of the determinant corresponding to the characteristic periods of the three like atoms from each molecule will be nearly equal. Let $\nu_1^2/a_1 = \tau_1$, $\nu_2^2/a_2 = \tau_2$, etc. Then we have to solve the determinantal equation

$$0 = \begin{vmatrix} \tau_1\left(\frac{1}{\nu_1^2} - \frac{1}{n^2}\right), & \frac{1}{\gamma_{1,1'}}, & \dots\dots & \frac{1}{\gamma_{1,3k}} \\ \frac{1}{\gamma_{1,1'}}, & \tau_2\left(\frac{1}{\nu_{1'}^2} - \frac{1}{n^2}\right), & \dots\dots & \frac{1}{\gamma_{2,3k}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{1}{\gamma_{1,3k}}, & \frac{1}{\gamma_{2,3k}}, & & \tau_{3k}\left(\frac{1}{\nu_{3k}^2} - \frac{1}{n^2}\right) \end{vmatrix}.$$

* See on this point W. Ramsay, Phil. Mag., 1887, p. 129, and his reference to van der Waal's views on the subject.

Here, if ν_1, ν_1', ν_1'' are the three equal constants for three like atoms in the three separate molecules, we require to find the values of n which differ only by a small quantity from ν_1. Let $1/\nu_1^2 - 1/n^2 = \mu$. Then, neglecting μ^4, we have to find μ from the equation

$$\begin{vmatrix} \tau_1\mu, & 1/\gamma_{1,1'}, & \dots & \dots & 1/\gamma_{1,3k} \\ 1/\gamma_{1,1'}, & \tau_{1'}\mu, & \dots & \dots & 1/\gamma_{2,3k} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1/\gamma_{1,3k-1}, & 1/\gamma_{2,3k-1}, & \dots \tau_{3k-1}\left(\frac{1}{\nu_{3k-1}^2} - \frac{1}{\nu_1^2}\right), & & 1/\gamma_{3k-1,3k} \\ 1/\gamma_{1,3k}, & 1/\gamma_{2,3k}, & \dots 1/\gamma_{3k-1,3k}, & & \tau_{3k}\left(\frac{1}{\nu_{3k}^2} - \frac{1}{\nu_1^2}\right) \end{vmatrix} = 0. \tag{18}$$

The problem thus reduces to the following one in pure mathematics: To expand the following *symmetrical* determinant in powers of small quantities, where each constituent $a_{r,s}$ is small if r and s are different, but only $a_{n,n}$, $a_{n-1,n-1}$; $a_{n-2,n-2}$ of the diagonal constituents are also small,

$$\Delta = \begin{vmatrix} a_{n,n}, & a_{n,n-1}, & \dots & a_{n,1} \\ a_{n-1,n}, & a_{n-1,n-1}, & \dots & a_{n-1,1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{1,n}, & a_{1,n-1}, & & a_{1,1} \end{vmatrix}. \tag{19}$$

Now

$$\Delta = D_1 a_{n,n} - \sum_{r}^{1 \text{ to } n-1} a_{n,r}^2 \beta_{r,r} - 2\sum_{rs}^{1 \text{ to } n-1} a_{n,r} a_{n,s} \beta_{r,s}, \tag{20}$$

where D_1 is the determinant obtained by suppressing the first column and row and $\beta_{r,s}$ is any minor of D_1; r being different from s and all values from 1 to $n-1$ being given to them.

Similarly if D_2 be obtained from D_1 as D_1 from Δ and so on, we have

$$D_1 = D_2 a_{n-1,n-1} - \sum_{r}^{1 \text{ to } n-2} a_{n-1,r}^2 \beta'_{r,r} - 2\sum_{rs}^{1 \text{ to } n-2} a_{n-1,r} a_{n-1,s} \beta'_{r,s},$$

$$D_2 = D_3 a_{n-2,n-2} - \sum_{r}^{1 \text{ to } n-3} a_{n-2,r}^2 \beta''_{r,r} - 2\sum_{rs}^{1 \text{ to } n-3} a_{n-2,r} a_{n-2,s} \beta''_{r,s}.$$

Now D_3 will occur in Δ multiplied by $a_{n,n} a_{n-1,n-1} a_{n-2,n-2}$ or $\tau_1 \tau_{1'} \tau_{1''} \mu^3$; that is to say, if we keep only terms of the order μ^3 we may put for D_3 its diagonal $a_{n-3,n-3} \dots a_{1,1}$, which is the one term in it containing no further small quantities; we thus find to our order of approximation,

$$D_2 = a_{n-2,\,n-2}a_{n-3,\,n-3}\ldots a_{1,1} - \sum_r^{1\text{ to }n-3} a^2_{n-2,\,r}\beta''_{r,\,r} - 2\sum_{rs}^{1\text{ to }n-3} a_{n-2,\,r}a_{n-2,\,s}\beta''_{r,\,s}.$$

For brevity we shall write $a_{r,\,r}\ldots a_{1,1} = P(r,\,r)$. Now it is easy to see that the most important term in $\beta''_{r,\,s} = (-1)^{r+s}\dfrac{P(n-3,\,n-3)}{a_{r,\,r}a_{s,\,s}} \times a_{r,\,s}$ and thus the last summation appears in D_2 with terms of the cubic order of small quantities or in Δ with those of the fifth order, hence we neglect it.

Further, $\beta''_{r,\,r} = \dfrac{P(n-3,\,n-3)}{a_{r,\,r}}$ to our degree of approximation, thus finally,

$$D_2 = P(n-2,\,n-2) - P(n-3,\,n-3)\sum_r^{1\text{ to }n-3}\frac{a^2_{n-2,\,r}}{a_{r,\,r}}.$$

We now turn to D_1 whose value must be taken to one degree higher approximation than D_2, we find by similar reasoning

$$\begin{aligned} D_1 = D_2 \times a_{n-1,\,n-1} &- \sum_r^{1\text{ to }n-3} a^2_{n-1,\,r}\frac{P(n-3,\,n-3)}{a_{r,\,r}} \times a_{n-2,\,n-2} \\ &- a^2_{n-1,\,n-2}P(n-3,\,n-3) - 2(-1)^{n-3-s}\sum_s^{1\text{ to }n-3} a_{n-1,\,n-2}a_{n-1,\,s} \\ &\qquad \times a_{n-2,\,s}\frac{P(n-3,\,n-3)}{a_{s,\,s}} \\ = P(n-3,\,n-3)\Bigg\{ & a_{n-1,\,n-1}a_{n-2,\,n-2} - a_{n-1,\,n-1}\sum_r \frac{a^2_{n-2,\,r}}{a_{r,\,r}} - a^2_{n-1,\,n-2} \\ &- a_{n-2,\,n-2}\sum_r^{1\text{ to }n-3}\frac{a^2_{n-1,\,r}}{a_{r,\,r}} - 2(-1)^{n-3-s}a_{n-1,\,n-2}\sum_s^{1\text{ to }n-3}\frac{a_{n-1,\,s}a_{n-2,\,s}}{a_{s,\,s}}\Bigg\}. \end{aligned}$$

Finally we obtain in like manner

$$\begin{aligned} D = D_1 \times a_{n,\,n} &- \sum_r^{1\text{ to }n-3}\frac{a^2_{n,\,r}}{a_{r,\,r}}P(n-3,\,n-3)\,a_{n-1,\,n-1}a_{n-2,\,n-2} \\ &- a^2_{n,\,n-1}a_{n-2,\,n-2}P(n-3,\,n-3) \\ &- a^2_{n,\,n-2}a_{n-1,\,n-1}P(n-3,\,n-3) \\ &+ 2a_{n,\,n-1}a_{n,\,n-2}a_{n-1,\,n-2}P(n-3,\,n-3) \\ &- 2(-1)^{n-3-s}\sum_s^{1\text{ to }n-3}\frac{a_{n,\,n-1}a_{n,\,s}a_{n-1,\,s}}{a_{s,\,s}}P(n-3,\,n-3)\,a_{n-2,\,n-2} \\ &- 2(-1)^{n-3-s}\sum_s^{1\text{ to }n-3}\frac{a_{n,\,n-2}a_{n,\,s}a_{n-2,\,s}}{a_{s,\,s}}P(n-3,\,n-3)\,a_{n-1,\,n-1}. \end{aligned}$$

Thus we obtain for Δ the value

$$\frac{\Delta}{P(n-3,\, n-3)} = a_{n,n}a_{n-2,n-2}\,a_{n-1,n-1} - a_{n,n}a^2_{n-1,n-2} - a_{n-1,n-1}a^2_{n,n-2}$$

$$- a_{n-2,n-2}a^2_{n,n-1} + 2a_{n,n-1}\,a_{n,n-2}\,a_{n-1,n-2}$$

$$- a_{n-1,n-1}a_{n-2,n-2}\sum_r^{1 \text{ to } n-3}\frac{a^2_{n,r}}{a_{r,r}}$$

$$- a_{n,n}\,a_{n-1,n-1}\sum_r^{1 \text{ to } n-3}\frac{a^2_{n-2,r}}{a_{r,r}}$$

$$- a_{n-2,n-2}a_{n,n}\sum_r^{1 \text{ to } n-3}\frac{a^2_{n-1,r}}{a_{r,r}}$$

$$- 2(-1)^{n-3-s}a_{n,n}\,a_{n-1,n-2}\sum_s^{1 \text{ to } n-3}\frac{a_{n-1,s}\,a_{n-2,s}}{a_{s,s}}$$

$$- 2(-1)^{n-3-s}a_{n-1,n-1}a_{n,n-2}\sum_s^{1 \text{ to } n-3}\frac{a_{n,s}\,a_{n-2,s}}{a_{s,s}}$$

$$- 2(-1)^{n-3-s}a_{n-2,n-2}\,a_{n,n-1}\sum_s^{1 \text{ to } n-3}\frac{a_{n,s}\,a_{n-1,s}}{a_{s,s}}\,.$$

13). Resuming the notation of our molecular problem, we have to find μ the cubic:

$$\tau_1\tau_{1'}\tau_{1''}\mu^3 - \mu^2\left\{\tau_{1'}\tau_{1''}\Sigma\frac{\tau_r^{-1}}{\gamma^2_{1r}}\left(\frac{1}{\nu_r^2}-\frac{1}{\nu_1^2}\right)^{-1} + \tau_1\tau_{1'}\Sigma\frac{\tau_r^{-1}}{\gamma^2_{1''r}}\left(\frac{1}{\nu_r^2}-\frac{1}{\nu_1^2}\right)^{-1}\right.$$

$$\left. + \tau_1\tau_{1''}\Sigma\frac{\tau_r^{-1}}{\gamma^2_{1'r}}\left(\frac{1}{\nu_r^2}-\frac{1}{\nu_1^2}\right)^{-1}\right\}$$

$$-\mu\left\{\tau_1\frac{1}{\gamma^2_{1'1''}} + \tau_{1'}\frac{1}{\gamma^2_{11''}} + \tau_{1''}\frac{1}{\gamma^2_{11'}}\right.$$

$$+ 2(-1)^p\frac{\tau_1}{\gamma_{1'1''}}\Sigma\frac{\tau_r}{\gamma_{1'r}\gamma_{1''r}}\left(\frac{1}{\nu_r^2}-\frac{1}{\nu_1^2}\right)^{-1}$$

$$+ 2(-1)^p\frac{\tau_1}{\gamma_{11''}}\Sigma\frac{\tau_r}{\gamma_{1r}\gamma_{1''r}}\left(\frac{1}{\nu_r^2}-\frac{1}{\nu_1^2}\right)^{-1}$$

$$\left. + 2(-1)^p\frac{\tau_{1''}}{\gamma_{11'}}\Sigma\frac{\tau_r}{\gamma_{1'r}\gamma_{1r}}\left(\frac{1}{\nu_r^2}-\frac{1}{\nu_1^2}\right)^{-1}\right\}$$

$$+ 2\frac{1}{\gamma_{11'}\gamma_{11''}\gamma_{1'1''}} = 0.$$

Here the summations are to extend over all the atoms of all three molecules except the three equal atoms, one in each molecule which have $2\pi/\nu_1$ for their characteristic free period, and p equals the number of places from the sth atom to the 2[d] in the series

$$1,\ 1',\ 1'',\ 2,\ 2'',\ 2'',\ \ldots\ s,\ s',\ s'',\ \ldots\ k,\ k',\ k''.$$

Remembering the value of τ_r, we may rewrite this equation,

$$\left.\begin{aligned} &\mu^3 - \mu^2\Sigma\left\{\frac{1}{\nu_1^2-\nu_r^2}\left(\frac{a_1a_r}{\gamma_{1r}^2}+\frac{a_{1'}a_r}{\gamma_{1'r}^2}+\frac{a_{1''}a_r}{\gamma_{1''r}^2}\right)\right\} \\ &\quad -\mu\left\{\frac{1}{\nu_1^4}\left(\frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2}+\frac{a_1a_{1''}}{\gamma_{11''}^2}+\frac{a_1a_{1'}}{\gamma_{11'}^2}\right)\right. \\ &\qquad\qquad + 2\Sigma(-1)^p\left(\frac{a_{1'}a_{1''}a_r}{\gamma_{1'1''}\gamma_{1'r}\gamma_{1''r}}+\frac{a_1a_{1''}a_r}{\gamma_{11''}\gamma_{1r}\gamma_{1''r}}+\frac{a_1a_{1'}a_r}{\gamma_{11'}\gamma_{1r}\gamma_{1'r}}\right) \\ &\qquad\qquad\qquad\qquad \left.\times\frac{1}{\nu_1^2(\nu_1^2-\nu_r^2)}\right\} \\ &\qquad\qquad + \frac{2}{\nu_1^6}\ \frac{a_1a_{1'}a_{1''}}{\gamma_{1'1''}\gamma_{11''}\gamma_{11'}} \end{aligned}\right\} = 0. \qquad (21)$$

As a verification of this equation, suppose each molecule to consist of one atom only, then the sum terms of course disappear and the equation agrees with that for the mutual influence of three equal atoms given on p. 86 (γ) of my first paper.

Suppose, on the other hand, that the molecules are so far apart that the vibrations of any atom depend only on the atoms of its own group. Then our cubic equation ought, by our Art. 9, equation (14), to reduce to

$$\left(\mu-\sum_1\frac{a_1a_s}{\gamma_{1s}^2}\ \frac{1}{\nu_1^2-\nu_s^2}\right)\left(\mu-\sum_2\frac{a_{1'}a_{s'}}{\gamma_{1's'}^2}\ \frac{1}{\nu_{1'}^2-\nu_{s'}^2}\right)\left(\mu-\sum_3\frac{a_{1''}a_{s''}}{\gamma_{1''s''}^2}\ \frac{1}{\nu_{1''}^2-\nu_{s''}^2}\right)=0. \qquad (22)$$

Now (21) agrees with this if we neglect all terms depending on *molecular* distances, so far as the first two terms are concerned, i. e. we have

$$\mu^3-\mu^2\left\{\sum_1\frac{a_1a_s}{\gamma_{1s}^2}\ \frac{1}{\nu_1^2-\nu_s^2}+\sum_2\frac{a_{1'}a_{s'}}{\gamma_{1's'}^2}\ \frac{1}{\nu_{1'}^2-\nu_{s'}^2}+\sum_3\frac{a_{1''}a_{s''}}{\gamma_{1''s''}^2}\ \frac{1}{\nu_{1''}^2-\nu_{s''}^2}\right\}=0.$$

Nor should we expect it to go further because we have neglected all powers of a/γ above the third, or above the third and its product into μ, whereas the full expansion of (22) contains the sixth power of a/γ. Now it would be troublesome to expand the determinant (14) up to sixth powers of a_{rs}. But (21) will readily enable us to ascertain what effect the presence of the other two molecules has

on the vibrations of the first. This arises from the fact that equation (21) is a true approximation to the value of μ up to third powers of *inter-molecular* (as distinguished from *inter-atomic*) distances, and thus we have only to take equation (22) for μ and vary its constants by adding the inter-molecular terms from (21). Suppose that (22) can be written

$$\mu^3 - \lambda_1\mu^2 + \lambda_2\mu - \lambda_3 = 0. \tag{23}$$

Then the variations in this cubic equation due to inter-molecular action are expressed as follows:

$$\left.\begin{aligned}
&\mu^3 - \mu^2\left\{\lambda_1 + \sum_{2,3}\frac{a_1a_r}{\gamma_{1r}^2} + \frac{1}{\nu_1^2-\nu_r^2} + \sum_{1,3}\frac{a_{1'}a_r}{\gamma_{1'r}^2}\,\frac{1}{\nu_{1'}^2-\nu_r^2}\right.\\
&\qquad\qquad\left. + \sum_{2,1}\frac{a_{1''}a_r}{\gamma_{1''r}^2}\,\frac{1}{\nu_{1''}^2-\nu_r^2}\right\}\\
&\quad + \mu\left\{\lambda_2 - \frac{1}{\nu_1^4}\left(\frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2} + \frac{a_1a_{1''}}{\gamma_{11''}^2} + \frac{a_1a_{1'}}{\gamma_{11'}^2}\right)\right.\\
&\qquad\qquad - 2\sum_{1,2,3}(-1)^p\left(\frac{a_{1'}a_{1''}a_r}{\gamma_{1'1''}\gamma_{1'r}\gamma_{1''r}} + \frac{a_1a_{1''}a_r}{\gamma_{11''}\gamma_{1r}\gamma_{1''r}} + \frac{a_1a_{1'}a_r}{\gamma_{11'}\gamma_{1r}\gamma_{1'r}}\right)\\
&\qquad\qquad\qquad\qquad\left.\times\frac{1}{\nu_1^2(\nu_1^2-\nu_r^2)}\right\}\\
&\quad - \left\{\lambda_3 - \frac{2}{\nu_1^6}\,\frac{a_1a_{1'}a_{1''}}{\gamma_{1'1''}\gamma_{11''}\gamma_{11'}}\right\}
\end{aligned}\right\} = 0. \tag{24}$$

Here r represents any atom of the three molecules, except the three $1, 1', 1''$ or which μ gives the period, and $\sum\limits_{2,3}$ means a summation with regard to r of all atoms in both the 2[d] and 3[d] molecules except $1'$ and $1''$; $\sum\limits_{1,2,3}$ means a summation with regard to r for all atoms in the 1[st], 2[d] and 3[d] molecules except 1, $1'$ and $1''$, and so on.

We shall denote the variations in the value of the coefficients of equation (23) as given in equation (24) by $\delta\lambda_1$, $\delta\lambda_2$ and $\delta\lambda_3$ respectively, thus we have to find μ from the equation

$$\mu^3 - \mu^2(\lambda_1 + \delta\lambda_1) + \mu(\lambda_2 + \delta\lambda_2) - (\lambda_3 + \delta\lambda_3) = 0,$$

or

$$\mu^3 - \mu^2\lambda_1 + \mu\lambda_2 - \lambda_3 - \mu^2\delta\lambda_1 + \mu\delta\lambda_2 - \delta\lambda_3 = 0.$$

Now for the mean sets of molecules the roots of equation (23) must all be equal, for they give the vibration of the one atom of three different molecules supposed to

be uninfluenced by each other, but in the same physical condition, i. e. the mean value of γ_{rs} = mean value of $\gamma_{r's'}$ = mean value of $\gamma_{r''s''}$, where γ_{rs} is the distance between r^{th} and s^{th} atom in the first molecule and $\gamma_{r's'}$, $\gamma_{r''s''}$ are like quantities for the other two molecules. Hence equation (22) has three equal roots $= \mu_1$, say. Thus the last equation may be written

$$(\mu - \mu_1)^3 - \mu^2\delta\lambda_1 + \mu\delta\lambda_2 - \delta\lambda_3 = 0.$$

Let μ_1 be changed owing to the changes in the coefficients to $\mu_1 + \delta\mu_1$. Then we find

$$(\delta\mu_1)^3 - \delta\mu_1^2\delta\lambda_1 - \delta\mu_1(2\mu_1\delta\lambda_1 - \delta\lambda_2) - \mu_1^2\delta\lambda_1 + \mu_1\delta\lambda_2 - \delta\lambda_3 = 0. \tag{25}$$

This is a cubic equation to find the three values of $\delta\mu_1$. Let us consider the order of the various terms of this equation, and suppose that to represent a quantity by $[1/a]^i[1/m]^j$ denotes that it is of the ith order in the inverse of inter-atomic distance (i. e. distance between atoms of the same molecule), and that it is of the jth order in the inverse of inter-molecular distance (i. e. in the distance between atoms of different molecules). Thus we find—

that	μ is represented by			$[1/a]^2$,
"	$\delta\lambda_1$	"	"	$[1/m]^2$,
"	$\delta\lambda_3$	"	"	$[1/m]^3$,
"	$\delta\lambda_2$	"	"	$[1/m]^2 + [1/m]^3 + [1/m]^2[1/a]$.

Hence we see that as a *first approximation* we may neglect $(\delta\mu_1)^2\delta\lambda_1$ and $2\mu_1\delta\mu_1\delta\lambda_1$ as compared with $\delta\mu_1\delta\lambda_2$; and further, $\mu_1^2\delta\lambda_1$ as compared with the terms in $\mu_1\delta\lambda_2$ of the order $[1/a]^2[1/m]^2$. Whether we can neglect $\delta\lambda_3$ as compared with $\mu_1\delta\lambda_2$ will depend on whether terms of order $[1/m]$ are comparable or not with terms of the order $[1/a]^2$. If the inverse of inter-molecular distance is only of the second order of small quantities as compared with the inverse of inter-atomic distance, then $\delta\lambda_3$ will have to be retained.

Thus we have to a first approximation for $\delta\mu$ the equation

$$\left.\begin{aligned}(\delta\mu_1)^3 - \frac{\delta\mu_1}{\nu_1^4}\left(\frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2} + \frac{a_1a_{1''}}{\gamma_{11''}^2} + \frac{a_1a_{1'}}{\gamma_{11'}^2}\right) + \frac{2}{\nu_1^6}\ \frac{a_1a_{1'}a_{1''}}{\gamma_{1'1''}\gamma_{11''}\gamma_{11'}} \\ - \frac{\mu_1}{\nu_1^4}\left(\frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2} + \frac{a_1a_{1''}}{\gamma_{11''}^2} + \frac{a_1a_{1'}}{\gamma_{11'}^2}\right)\end{aligned}\right\} = 0. \tag{26}$$

14). From this equation we can draw some important conclusions:

a). The modified period of an atom in one molecule owing to the action

of other molecules in its neighborhood, is to a first approximation a function only of the relative positions of the *corresponding* atoms in the modifying molecules. Thus the vibrations of the atom A will depend not only on its distances from its kindred atoms A' and A'' in other molecules, but also on the distance of A' from A''; it will not, however (to a first approximation), depend upon A's distance from B' or C'', etc.

b). Suppose inter-molecular distance is enormously greater than inter-atomic distance, or $[1/m]$ very small as compared even with $[1/a]^2$. Then we neglect the first term in the constant of the equation for $\delta\mu_1$ as compared with the second. This shows us that $\delta\mu_1$ will be of the order $[1/a]^{\frac{2}{3}}[1/m]^{\frac{2}{3}}$, hence since $[1/m]$ is negligible as compared with $[1/a]^2$, it follows that the term involving $\delta\mu_1$ may be neglected as compared with $(\delta\mu_1)^3$, or we have

$$\delta\mu_1 = \left\{\frac{\mu_1}{\nu_1^4}\left(\frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2} + \frac{a_1 a_{1''}}{\gamma_{11''}^2} + \frac{a_1 a_{1'}}{\gamma_{11'}^2}\right)\right\}^{\frac{1}{3}}. \tag{27}$$

In this case there is only *one real* value of $\delta\mu_1$, the same for all three kin-atoms of the three molecules. Thus we should expect, if the molecules be so far apart that they have no modifying action and so exhibit a pure bright line spectrum, that when they are brought closer together there would first be a slight shifting of their bright lines, and this shifting—since $\delta\mu_1$ is positive, denoting a decrease in the period—is towards the violet end of the spectrum. So soon, however, as the effect of pressure or temperature causes $[1/m]$ to be comparable with $[1/a]^2$, the individual bright line is replaced by three others, if only two molecules have modifying influence on a third, but by $(p+1)$ bright lines if p molecules exercise such influence. Here we see, perhaps, a little more clearly than in the first paper, how the bright line spectrum becomes fluted and ultimately continuous.

In this case the variation in μ_1 due to the modifying action of the kindred atoms in other molecules, is of the order $[1/a]^{\frac{2}{3}}[1/m]^{\frac{2}{3}}$, or its ratio to μ_1 of the order $(a/m)^{\frac{2}{3}}$. That is, when inter-molecular distance is very great as compared with inter-atomic, the modification in period of any atom is of the order

$$\left(\frac{\text{intera-tomic distance}}{\text{inter-molecular distance}}\right)^{\frac{2}{3}}.$$

c). Suppose $[1/m]$ of the order $[1/a]^2$. Then we must retain all terms of the equation (26). We see that $\delta\mu_1$ will be of the order $[1/a]^2$, but this is pre-

cisely the same order as μ_1 itself, or we conclude: That modified action becomes as important as inter-atomic action in settling the period of an atom if the inverse of inter-molecular distance is comparable with the inverse square of inter-atomic distance. The reason of this apparent anomaly lies in the fact that an atom will produce greater effect on a kindred atom when at a considerably greater distance than on non-kindred atoms at a considerably less distance. In deducing this result we have supposed that there are *not* two equal or kindred atoms in any one molecule. If there were, the quantity μ_1 is not of the order $[1/a]^2$, but of the order $[1/a]$, and is therefore sensibly greater than the modifying effect of kindred atoms in other molecules.

d). If $[1/m]$ be sensibly greater than $[1/a]^2$, then the equation for $\delta\mu_1$ reduces to that for the mutual influence of three equal atoms,* and the variation in μ_1 is simply got by adding to the value of μ_1 obtained from the isolated molecule, the value of μ_1 which would be obtained from treating the three kindred atoms as isolated in space.

With regard to all the above cases we must remark that inter-molecular action is accumulative, i. e. a great number of molecules may be near enough to affect the period of vibration of an atom of the molecule under consideration, and we may have to sum a very great number of small terms, so increasing the total effect.

According to Ampère and Becquerel, inter-molecular distance is enormous as compared with molecular diameter; according to Babinet, it is at least as 1800:1. Sir W. Thomson, however, considers that the mean distance of two contiguous molecules of a solid is less than $\frac{1}{100,000,000}$ of a centimeter, while the diameter of a *gaseous* molecule is greater than $\frac{1}{500,000,000}$ of a centimeter. Supposing the diameter of a molecule not to differ very much in the solid and gaseous conditions, this would lead to inter-molecular distance being less than five times inter-atomic distance, or to the influence of molecules on the atomic vibrations of each other being very considerable.† Although it is not certain that the ratio is so small as this, still the phenomena of continuous and fluted spectra, as well as the possibility of multi-constant elasticity, lead us

* See p. 86 (γ) of my first paper.

† See Thomson and Tait: *Natural Philosophy*, Part II, p. 502, and Todhunter and Pearson: *History of Elasticity*, Vol. II, p. 184.

to believe that inter-molecular action on atomic vibrations is very sensible, and therefore to adopt what Jellett has termed the *Hypothesis of modified Action.*

e). It may not be without interest to mark the exact effect of non-kindred atoms in other molecules on the atomic vibrations of a given molecule.

Let μ' be a root of equation (24) when we neglect the modifying action of non-kindred atoms, so that $\mu_1' = \mu_1 + \delta\mu_1$. Thus we may write

$$\mu_1'^3 - \mu_1'^2\lambda_1' + \mu_1'\lambda_2' - \lambda_3' = 0,$$

where the dashed λ's are the λ's of equation (23) affected only by the modifying action of kindred atoms. Now, owing to the action of the non-kindred atoms, λ_1' becomes $\lambda_1' + \delta\lambda_1'$ and λ_2' becomes $\lambda_2' + \delta\lambda_2'$. The effect on μ_1', which we will represent by $\delta\mu_1'$, is easily found to be

$$\delta\mu_1' = \frac{\mu_1'(\mu_1'\delta\lambda_1' - \delta\lambda_2')}{3\mu_1'^2 - 2\mu_1'\lambda_1' + \lambda_2'}.$$

The denominator $= 3(\mu_1 + \delta\mu_1)^2 - 2(\mu_1 + \delta\mu_1)(\lambda_1 + \delta\lambda_1) + \lambda_2 + \delta\lambda_2$, but $3\mu_1^2 - 2\mu_1\lambda_1 + \lambda_2 = 0$ and $\lambda_1 = 3\mu_1$. Hence we have the denominator $= 3(\delta\mu_1)^2 - 2\mu_1\delta\lambda_1 - 2\delta\mu_1\delta\lambda_1 + \delta\lambda_2$. Considering the order of terms, we see that the two mid-terms are small as compared with the extreme terms, and thus we may take the denominator $= 3(\delta\mu_1)^2 + \delta\lambda_2$. Turning to the numerator, we have

$\mu_1'\delta\lambda_1'$ represented by $[1/a]^2[1/m]^2$

and $\delta\lambda_2'$ " " $[1/m]^3 + [1/m]^2[1/a]$.

Thus the second part of the latter is the most important, or we may write

$$\delta\mu_1' = \frac{-\mu_1'\delta\lambda_2'}{3(\delta\mu_1)^2 + \delta\lambda_2}. \tag{28}$$

We ought to distinguish two cases—

(i). $[1/m]$ is immensely greater than $[1/a]$ and not comparable even with $[1/a]^2$.

Hence $\delta\lambda_2$ may be neglected as compared with $\delta\mu_1$ which is now given by equation (27); further, μ_1' may be replaced by μ_1 and

$$\delta\lambda_2' = -\frac{2}{\nu_1^2}\left\{\sum_{23}(-1)^p \frac{a_{1'}a_{1''}a_r}{\gamma_{1'1''}\gamma_{1'r}\gamma_{1''r}}\frac{1}{(\nu_1^2 - \nu_r^2)} + \sum_{1,3}(-1)^p \frac{a_1 a_{1''}a_r}{\gamma_{11''}\gamma_{1r}\gamma_{1''r}}\frac{1}{(\nu_1^2 - \nu_r^2)} + \sum_{1,2}(-1)^p \frac{a_1 a_{1'}a_r}{\gamma_{11'}\gamma_{1r}\gamma_{1'r}}\frac{1}{(\nu_1^2 - \nu_r^2)}\right\}. \tag{29}$$

Thus we see that the order of $\delta\mu_1'$ is represented by

$$\frac{[1/a]^3[1/m]^2}{[1/a]^{\frac{4}{3}}[1/m]^{\frac{4}{3}}} = [1/a]^{\frac{5}{3}}[1/m]^{\frac{2}{3}}.$$

Thus the ratio of $\delta\mu_1'$ to $\delta\mu_1$ is of the order $[1/a]^2$, or if $\delta\mu_1$ be of the second order of small quantities, this modification is of the *fourth order.*

The character of the disturbance in μ_1 as given by $\delta\lambda_2'$ is noteworthy. It does not involve the inter-molecular distances of kindred atoms other than those the variation of the period of which we are dealing with. Thus if A, B, C, ... A', B', C', ... A'', B'', C'', ... denote the atoms of the three molecules, A, A', A'' being kindred atoms, then the manner in which B', B'', C', C'' modify the period of A is by their distances from A, A' and A'' and not by their distances from B or C, or even from B' and C'; the latter are distances which it would seem probable might have occurred.

Case (ii). $[1/m]$ is of the same order as $[1/a]^2$.

Here the term of the order $[1/m]^2$ in $\delta\lambda_2$ must be retained in the denominator and $\mu_1 + \delta\mu_1$ must be put for μ_1' in the numerator. The order of $\delta\mu_1'$ will now be represented by $[1/a]^3$, or the ratio of $\delta\mu_1'$ to $\delta\mu_1$, or to μ_1, is only of the order $[1/a]$, thus its importance has much increased owing to the reduction in molecular distance. The exact value of $\delta\mu_1'$ may be easily written down from equations (28) and (29).

The general conclusion we seem compelled to form is, that the period of atomic vibrations will be sensibly modified by inter-molecular action, so soon as $[1/m]$ is not negligible as compared with $[1/a]^2$.

15). We pass now from the consideration of the modifying action on period to the modifying action on amplitude. Let C_s denote the amplitude of the term of period $2\pi/n$ in the sth atom. It will be sufficient to determine all these amplitudes if we investigate types of each period. We shall endeavor, then, to find the relation between C_1, C_2, $C_{1'}$ and $C_{2'}$, when $n =$ one of the values found above for $\mu_1 + \delta\mu_1 + \delta\mu_1'$.

Let $\alpha_{rs} =$ the minor of the determinant obtained by the rth row and s^{th} column of the form in equation (18); then by the usual theory for the solution of equations of the type (11), we have

$$\frac{C_1}{\alpha_{11}} = \frac{C_{1'}}{\alpha_{1'1}} = \frac{C_2}{\alpha_{12}} = \frac{C_{2'}}{\alpha_{12'}}.$$

We have now to find approximate values for these minors. Let us take them in order.

We have $\alpha_{11} = D_1$ of our Art. 12, but it will not do to take for its value simply the expression there cited in terms of the elements, because that expression only gives it so far as the cubes of small elements, our object being then merely to find how the terms of a known cubic were affected by inter-molecular distances up to the cubes. We have now to calculate out D_1 as far as the fourth powers of inter-atomic and the square powers of inter-molecular distance. Now

$$D_1 = \alpha_{11} = \begin{vmatrix} T_{1'} & a_{1'1''} & a_{1'2} & a_{1'2'} & a_{1'2''} \dots \\ a_{1''1'} & T_{1''} & a_{1''2} & a_{1''2'} & a_{1''2''} \dots \\ a_{21'} & a_{21''} & T_2 & a_{22'} & a_{22''} \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

where $T_r = \tau_r \left(\frac{1}{\nu_r^2} - \frac{1}{n^2}\right)$ and $a_{rs} = \frac{1}{\gamma_{rs}}$. Here $T_{1'}$ and $T_{1''}$ are of the order $[1/a]^2$, but T_r for all values except $1'$ and $1''$ is supposed not small. Let us pick out all terms of order equal or less than $[1/a]^4$. If we retain all the large T's we have the terms $T_2 \dots T_{k''} (T_{1'} T_{1''} - a_{1'1''}^2)$. Let us represent the product of all large T's by P, the above term is thus

$$P (T_{1'} T_{1''} - a_{1'1''}^2).$$

Now omit one large T, say T_r, and select (i) an inter-molecular term, say $a_{r'r}$, out of the rth column; this will have to be multiplied by $a_{r1'}$ and $T_{1''}$, it is thus of the order $[1/m]^2 [1/a]^2$ and to be neglected; (ii) an inter-atomic term; this is impossible. Hence T_r gives us no terms. Now consider $T_{r'}$; we may now take the inter-atomic term $a_{1'r'}$ and it must be multiplied by $a_{r'1'} T_{1''}$. Thus we have two expressions of the form

$$T_{1''} P \sum_2 \frac{a_{1'r'}^2}{T_{r'}} \text{ and } T_{1'} P \sum_3 \frac{a_{1''r''}^2}{T_{r''}},$$

and a little consideration shows that the signs of these terms are *negative.* They are all we get to the order $[1/a]^4$ by omitting one large T.

Now let us omit two large T's. As a first case say T_r and T_s corresponding to atoms both in the first molecule, then the quantities to be chosen out of the 1st, 2d, r^{th} and s^{th} columns must all be inter-atomic, but this is easily seen to be

impossible. As a second case take $T_{r'}$ and $T_{s'}$. We have to take two elements out of the first two rows; they cannot be taken out of the first two columns because these are of the order $[1/a]^2$ at least, therefore we must choose $a_{1'r'}$, but this throws us back on $a_{1''s'}$, which is inter-molecular. Hence no terms such as $T_{r'}$ and $T_{s'}$ can be omitted. Now let us omit $T_{r'}$ and $T_{s''}$, we are then compelled to take the inter-atomic terms $a^2_{r'1'}a^2_{s''2''}$. The sign of these terms is found to be positive, or we have the expression

$$P\Sigma \frac{a^2_{r'1'}\,a^2_{s''2''}}{T_{r'}\,T_{s''}}.$$

As a fourth case let us take out two terms like T_r and $T_{s'}$; then the choice of an inter-atomic term out of the rth column becomes impossible, and thus this form of term does not appear to our degree of approximation.

Finally we have

$$D_1 = P\left\{T_1 T_{1''} - T_{1''}\sum_2 \frac{a^2_{1'r'}}{T_{r'}} - T_{1'}\sum_3 \frac{a^2_{1''r''}}{T_{r''}} - a^2_{1'1''} + \Sigma \frac{a^2_{r'1'}\,a^2_{s''2''}}{T_{r'}\,T_{s''}}\right\}$$
$$= P\left\{\left(T_{1'} - \sum_3 \frac{a^2_{1''r''}}{T_{r''}}\right)\left(T_{1''} - \sum_2 \frac{a^2_{1'r'}}{T_{r'}}\right) - a^2_{1'1''}\right\}.$$

Thus $$\alpha_{11} = P\left\{\left(\tau_{1'}\mu - \sum_3 \frac{a_{r''}}{\gamma^2_{1''r''}}\,\frac{n^2}{n^2-\nu^2_r}\right)\left(\tau_{1''}\mu - \sum_2 \frac{a_{r'}}{\gamma^2_{1'r'}}\,\frac{n^2}{n^2-\nu^2_{r'}}\right) - \frac{1}{\gamma^2_{1'1''}}\right\}.$$

Now $$\sum_3 \frac{a_{r''}}{\gamma^2_{1''r''}}\,\frac{n^2}{n^2-\nu^2_r} = \sum_3 \frac{a_{r''}}{\gamma^2_{1''r''}}\,\frac{\nu^2_1}{\nu^2_1-\nu^2_r} = \tau_{1'}\mu_1,$$

if we neglect in the value of α_{11} terms of the order $(a/\gamma)^6$. Now $\mu = \mu_1 + \delta\mu_1$, hence

$$\alpha_{11} = P\tau_{1'}\tau_{1''}\left\{(\delta\mu_1)^2 - \frac{a_{1'}a_{1''}}{\nu^4_1\gamma^2_{1'1''}}\right\}. \tag{30}$$

Let us now find $\alpha_{1'1}$ to the same order of small quantities. In precisely the same manner we have

$$\left.\begin{aligned}\alpha_{1'1} &= -P\left\{a_{1'1}T_{1''} - \sum_3 \frac{a^2_{1''r''}a_{1'1}}{T_{r''}} - a^2_{1'1''}\right\}\\ &= -P\left\{\frac{\tau_{1''}(\mu-\mu_1)}{\gamma_{1'1}} - \frac{1}{\gamma^2_{1'1''}}\right\}\\ &= -\frac{P\tau_{1'}\tau_{1''}}{\nu^2_1}\left\{\frac{a_{1'}}{\gamma_{1'1}}\,\delta\mu_1 - \frac{1}{\nu^2_1}\,\frac{a_{1'}a_{1''}}{\gamma^2_{1'1''}}\right\}\end{aligned}\right\}. \tag{31}$$

Further we find

$$\left.\begin{aligned}\alpha_{12} &= -\frac{P\tau_{1'}\tau_{1''}}{\gamma_{12}\,T_2}\left\{(\delta\mu_1)^2 - \frac{a_{1'}a_{1''}}{\nu^4_1\gamma^2_{1'1''}}\right\}\\ &= -P\tau_{1'}\tau_{1''}\frac{\nu^2_1}{\nu^2_1-\nu^2_2}\;\frac{a_2}{\gamma_{12}}\left\{(\delta\mu_1)^2 - \frac{a_{1'}a_{1''}}{\nu^4_1\gamma^2_{1'1''}}\right\}\end{aligned}\right\}. \tag{32}$$

Finally, we have

$$\alpha_{12'} = P \frac{\tau_{1'}\tau_{1''}}{\nu_1^2 - \nu_{2'}^2} \frac{a_{2'}}{\gamma_{1'2'}} \left\{ \frac{a_{1'}}{\gamma_{11'}} \delta\mu_1 - \frac{a_{1'}a_{1''}}{\nu_1^2 \gamma_{1'1''}^2} \right\}. \tag{33}$$

We can now write down the values of $C_{1'}$, C_2, $C_{2'}$ in terms of C_1, we find

$$C_2 = - \frac{\nu_1^2}{\nu_1^2 - \nu_2^2} \frac{a_2}{\gamma_{12}} C_1. \tag{34}$$

Suppose $[1/m]$ not comparable with $[1/a]^2$, then

$$C_{1'} = - \frac{1}{\nu_1^2} \frac{a_{1'}}{\gamma_{11'}} \frac{1}{\delta\mu_1} C_1, \tag{35}$$

$$C_{2'} = \frac{1}{\nu_1^2 - \nu_{2'}^2} \frac{a_{2'}a_{1'}}{\gamma_{1'2'}\gamma_{11'}} \frac{1}{\delta\mu_1} C_1. \tag{36}$$

These results, although only approximations, seem to be of very considerable interest. I proceed to note some points connected with them.

a). Equation (34) is precisely the same equation as (15), determining the relation between the amplitudes of the 1st and 2d atoms of the same molecule, when no other molecule is affecting their vibrations. Thus it would seem that, to the approximation we have adopted, the 1st atom produces the same amplitudes in the atoms of its own molecule notwithstanding the presence of other molecules in the field. All we can safely say, however, is that the all-important or leading term in the amplitude remains unchanged. Other small terms may be added to it, for although we have calculated α_{11} and α_{12} to the order $[1/a]^4$, we have divided them both by a factor of the order $[1/a]^4$ when finding the ratio of C_1 to C_2. Hence to have obtained the modifying influence of the other molecules we ought to have calculated the minors to $[1/a]^6$. At the same time we shall see immediately that the influence of C_2 on $C_{2'}$ is exerted indirectly, i. e. by means of its influence on $C_{1'}$; hence it seems more than probable that the modifying influence of other molecules on C_2 (the amplitude corresponding to C_1) is exerted by altering C_1, or also indirectly.

b). Equation (35) shows us that when $[1/m]$ is not comparable with $[1/a]^2$, then since $\delta\mu_1$ is small, there is a very sensible amplitude $C_{1'}$ produced in the first atom of the second molecule by the action of the kindred atom in the first. By equation (27) $(\delta\mu_1)$ is of the order $[1/m]^{\frac{2}{3}}$. Hence $C_{1'}/C_1$ is of the order $[1/m]^{\frac{1}{3}}$, or *if* $[1/m]$ *be comparable only with* $[1/a]^3$, *still the effect of the first atom in the first molecule on its kindred atom in the second, will be as great as its effect on the other atoms of its own molecule, i. e. both will be of the order* $[1/a]$.

c). By equation (34) as type we should have

$$C_{2'} = -\frac{\nu_{1'}^2}{\nu_{1'}^2 - \nu_{2'}^2}\,\frac{a_{2'}}{\gamma_{1'2'}}\,C_{1'}.$$

This gives the effect of an amplitude $C_{1'}$ in the $1'^{\text{th}}$ atom on the amplitude of the $2'^{\text{d}}$ atom; or, if $C_{1'}$ be the variation in amplitude of the $1'^{\text{th}}$ atom due to the 1st atom of the first molecule, we have by (35):

$$C_{2'} = -\frac{\nu_{1'}^2}{\nu_{1'}^2 - \nu_{2'}^2}\,\frac{a_{2'}}{\gamma_{1'2'}}\left(-\frac{1}{\nu_1^2}\,\frac{a_{1'}}{\gamma_{11'}}\,\frac{1}{\delta\mu_1}\,C_1\right) = \frac{1}{\nu_1^2 - \nu_{2'}^2}\,\frac{a_{2'}a_{1'}}{\gamma_{1'2'}\gamma_{11'}}\,\frac{1}{\delta\mu_1}\,C_1,$$

which is exactly the value given by (36). This proves the point I have referred to above, namely, that whatever influence non-kindred atoms in different molecules exert on each other is indirect and takes place through their effect on their kindred atoms. This of course is only true for the approximation to which we have gone. The effect is, indeed, of a higher order of small quantities than that of the 1st atom on the 2^{d} or on the $1'^{\text{th}}$. If $[1/m]$ is comparable with $[1/a]^3$ it will be of the $[1/a]^2$ order, while the latter are of the $[1/a]$ order.

We may conclude, then, that when molecular distance is such that $[1/m]$ cannot be compared with $[1/a]^2$, then the modifying action of two molecules on each other's atomic vibrations is practically limited to the mutual action of their kindred atoms.

Now let us suppose $[1/m]$ is comparable with $[1/a]$. What we have said above about equation (34) still holds, but we find for $C_{1'}$ and $C_{2'}$

$$C_{1'} = -\frac{\dfrac{a_{1'}}{\gamma_{11'}}\,\dfrac{\delta\mu_1}{\nu_1^2} - \dfrac{1}{\nu_1^4}\,\dfrac{a_{1'}a_{1''}}{\gamma_{1'1''}^2}}{(\delta\mu_1)^2 - \dfrac{1}{\nu_1^4}\,\dfrac{a_{1'}a_{1''}}{\gamma_{1'1''}^2}}\,C_1, \tag{37}$$

$$C_{2'} = \frac{\nu_1^2}{\nu_1^2 - \nu_{2'}^2}\,\frac{a_{2'}}{\gamma_{1'2'}}\,\frac{\dfrac{a_{1'}}{\gamma_{11'}}\,\dfrac{\delta\mu_1}{\nu_1^2} - \dfrac{1}{\nu_1^4}\,\dfrac{a_{1'}a_{1''}}{\gamma_{1'1''}^2}}{(\delta\mu_1)^2 - \dfrac{1}{\nu_1^4}\,\dfrac{a_{1'}a_{1''}}{\gamma_{1'1''}^2}}\,C_1. \tag{38}$$

Now the numerator and denominator on the right-hand side of equation (37) are both of the same order, or we draw the conclusion that $C_{1'}$ and C_1 are of the same order. This follows from the fact that $\delta\mu_1$ is of the order $[1/a]^2$: see our

Art. 14 (c). Further, the effect of C_1 on $C_{2'}$ is as before indirect and due to the effect of C_1 on $C_{1'}$, but its order as compared with C_1 is now $[1/a]$. We are thus able to draw the following important conclusions for the case of $[1/m]$ being comparable with $[1/a]^2$:

a′). The variation in the amplitude of an atom due to the vibration of a kindred atom in another molecule is of the same order as the amplitude of the second atom.

b′). The variation in the amplitude of one atom due to the vibration of a non-kindred atom in another molecule is indirect, being produced by the influence exerted by the second atom on its kindred atom in the first molecule. It is of the same importance, however, as the variations produced in the amplitude of the same atom by any atom in its own molecule.

c′). The variations produced in the amplitudes of the atoms of one molecule by an atom of another are sensibly affected by the presence of a third molecule. This effect depends, in the approximation to which we have gone, entirely on the distance between those atoms of the first and third molecules which are kindred to the modifying atom of the second.

It should be noticed that owing to the value of $\delta\mu_1$, a remark similar to the last, (c′), may be made for the system (35) and (36), where $\delta\mu_1$ is determined by (27). The modifying action of the presence of the third molecule will not be so perceptible, however, as the variations in amplitude are so much less.

From the above statements flow the truth of the *hypothesis of modified action* and of the multi-constant equations of elasticity for matter built up of ether squirts. This will appear still more clearly if we determine the general form* of the inter-molecular force. To this we devote the following article.

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* I ought to note here that the above equations are susceptible of almost infinite variety. Their complexity becomes very considerable when, instead of only one atom of each kind in each molecule, we place p equal atoms in each k-atomic molecule—this being probably a very ordinary occurrence in nature. Further interesting results arise when we have two different atoms, whose periods are, however, nearly equal. In this case we have more diagonal terms of the fundamental determinant *small*, and the approximations must be carried still further. In fact it may be safely said that every substance, the atomic structure of the molecules of which is known, would require a mathematical dissertation to itself. Nor ought this complexity to discourage us. Whatever be the element of matter, we expect it to possess great simplicity of structure (and this is at least satisfied in the ether squirt), but this very simplicity has to explain the nigh infinite range of chemical and physical properties in each individual substance, or it must lead us to a mathematical theory of the substance of the very highest complexity in itself.

16). From the preceding article we find the following values for ϕ_1, $\phi_{1'}$ and ϕ_2 as types of vibration:

$$\left.\begin{aligned}\phi_1 = {} & C_1 \cos(n_1 t + \alpha_1) + \varkappa_{1'} C_{1'} \cos(n_{1'} t + \alpha_{1'}) + \varkappa_{1''} C_{1''} \cos(n_{1''} t + \alpha_{1''}) \\ & - \Sigma^1 \frac{\nu_r^2}{\nu_r^2 - \nu_1^2} \frac{a_1}{\gamma_{1r}} C_r \cos(n_r t + \alpha_r) \\ & + \Sigma_1^2 \lambda_{1,r'} C_{r'} \cos(n_{r'} t + \alpha_{r'}) \\ & + \Sigma_1^3 \lambda_{1,r''} C_{r''} \cos(n_{r''} t + \alpha_{r''})\end{aligned}\right\}, \quad (39)$$

$$\left.\begin{aligned}\phi_{1'} = {} & \varkappa_1' C_1 \cos(n_1 t + \alpha_1) + C_{1'} \cos(n_{1'} t + \alpha_{1'}) + \varkappa_{1''}' C_{1''} \cos(n_{1''} t + \alpha_{1''}) \\ & - \Sigma^2 \frac{\nu_{r'}^2}{\nu_{r'}^2 - \nu_{1'}^2} \frac{a_{1'}}{\gamma_{1'r'}} C_{r'} \cos(n_{r'} t + \alpha_{r'}) \\ & + \Sigma_2^1 \lambda_{1'r}' C_r \cos(n_r t + \alpha_r) \\ & + \Sigma_2^3 \lambda_{1'r''}' C_{r''} \cos(n_{r''} t + \alpha_{r''})\end{aligned}\right\}, \quad (40)$$

$$\left.\begin{aligned}\phi_2 = {} & C_2 \cos(n_2 t + \alpha_2) + \varkappa_{2'} C_{2'} \cos(n_{2'} t + \alpha_{2'}) + \varkappa_{2''} C_{2''} \cos(n_{2''} t + \alpha_{2''}) \\ & - \Sigma^1 \frac{\nu_r^2}{\nu_r^2 - \nu_2^2} \frac{a_2}{\gamma_{2r}} C_r \cos(n_r t + \alpha_r) \\ & + \Sigma_1^2 \lambda_{2,r'} C_{r'} \cos(n_{r'} t + \alpha_{r'}) \\ & + \Sigma_1^3 \lambda_{2,r''} C_{r''} \cos(n_{r''} t + \alpha_{r''})\end{aligned}\right\}, \quad (41)$$

where

$$\varkappa_{s'} = -\frac{\dfrac{a_s}{\gamma_{ss'}} \dfrac{\delta\mu_{s'}}{\nu_{s'}^2} - \dfrac{1}{\nu_{s'}^4} \dfrac{a_s a_{s''}}{\gamma_{ss''}^2}}{(\delta\mu_{s'})^2 - \dfrac{1}{\nu_{s'}^4} \dfrac{a_s a_{s''}}{\gamma_{ss''}^2}},$$

and $\varkappa_{s''}$ equals the value of $\varkappa_{s'}$ with the subscripts s' and s'' interchanged.

$$\varkappa_s' = -\frac{\dfrac{a_{s'}}{\gamma_{ss'}} \dfrac{\delta\mu_s}{\nu_s^2} - \dfrac{1}{\nu_s^4} \dfrac{a_{s'} a_{s''}}{\gamma_{s's''}^2}}{(\delta\mu_s)^2 - \dfrac{1}{\nu_s^4} \dfrac{a_{s'} a_{s''}}{\gamma_{s's''}^2}},$$

and $\varkappa_{s''}'$ equals the value of $\varkappa_s'$ with the subscripts s and s'' interchanged.

$$\lambda_{s,r'} = \frac{\nu_{r'}^2}{\nu_{r'}^2 - \nu_s^2} \frac{a_s}{\gamma_{rs}} \frac{\dfrac{a_r}{\gamma_{rr'}} \dfrac{\delta\mu_{r'}}{\nu_{r'}^2} - \dfrac{1}{\nu_{r'}^4} \dfrac{a_r a_{r''}}{\gamma_{rr''}^2}}{(\delta\mu_{r'})^2 - \dfrac{1}{\nu_{r'}^4} \dfrac{a_r a_{r''}}{\gamma_{rr''}^2}},$$

and $\lambda_{s,r''}$ is the same expression with r' changed to r'' and r'' to r'.

$$\lambda'_{s'r} = \frac{\nu_r^2}{\nu_{r'}^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{r's'}} \frac{\frac{a_{r'}}{\gamma_{r'r}} \frac{\delta\mu_r}{\nu_r^2} - \frac{1}{\nu_r^4} \frac{a_{r'} a_{r''}}{\gamma_{r'r''}^2}}{(\delta\mu_r)^2 - \frac{1}{\nu_r^4} \frac{a_{r'} a_{r''}}{\gamma_{r'r''}^2}},$$

and $\lambda'_{s',r''}$ is the same expression with the subscripts r and r'' interchanged.

Finally $\overset{1}{\Sigma}$ denotes a summation for all the atoms of the first molecule except those giving an infinite value to the subject of summation. $\overset{2}{\Sigma}$ has a like value for the second molecule. Further, Σ_2^1 denotes a summation of all values of r for the first molecule, r' in each term being given the corresponding value from the second molecule, Σ_2^3, Σ_1^2, etc., have like meanings; in all cases those values of r, r' or r'' being excluded which give rise to infinite terms.

The above results agree with those of my first paper if we put the $\varkappa$'s and λ's zero, or neglect the semi-chemical action of kindred atoms in producing inter-molecular force (see §42, p. 105, of that paper).

Now, in order to determine inter-molecular force, we must find the value of such terms as

$$\frac{1}{4\pi\rho} \frac{\dot{\phi}_1 \dot{\phi}_{1'}}{\gamma_{11'}} \text{ and } \frac{1}{4\pi\rho} \frac{\dot{\phi}_1 \dot{\phi}_{2'}}{\gamma_{12'}},$$

which are types of those occurring in the force function of equation (9). This involves ascertaining the mean values of such expressions as $\dot{\phi}_1 \dot{\phi}_{1'}$ and $\dot{\phi}_1 \dot{\phi}_{2'}$, and this will depend on the relative magnitude of inter-molecular and inter-atomic distances. We shall treat accordingly the following cases:

(i). Inter-molecular distance is quite incomparable with inter-atomic distance (i. e. $[1/m]^{\frac{1}{3}}$ is negligible as compared with $[1/a]$.

(ii). Inter-molecular distance is very great as compared with inter-atomic distance, but $[1/m]^{\frac{1}{3}}$ cannot be neglected as compared with $[1/a]$.

(iii). Inter-molecular distance is comparable with $[1/a]^2$.

These cases may be represented by

$$\begin{aligned} &\text{(i)} \quad [1/m] < [1/a]^3, \\ &\text{(ii)} \quad [1/m] = [1/a]^3, \\ &\text{(iii)} \quad [1/m] = [1/a]^2. \end{aligned}$$

If $[1/m] = [1/a]$, we obviously have complete chemical union between molecules, and therefore molecules cease to exist as such. This case we may exclude from our consideration.

Now, an important point still remains unsettled, namely: Is there any relation between the phases of the like vibrational terms in two like molecules? This point is only important for cases (i) and (ii), but for them it is of crucial importance. If two like molecules be at such distances as in no way to affect each other's periods, the vibrations characteristic of kin-atoms will be of the same period: will they also be of the same phase? In my first paper I defined molecular force to arise from the interaction of the *free* vibrations of the molecules. I did not suppose the molecules to form a single system with mutually *forced* vibrations, or inter-molecular force partially dependent on a semi-chemical action between kin-atoms of different molecules. Further investigation of the fundamental determinant has taught me the importance of this action at least for the case $[1/m] = [1/a]^2$.

Now the results of this assumption or rather omission in my first paper were the following:

(i). Supposing the phases of the like vibrations in molecules of the same substance to be unequal, then the first term in inter-molecular action is an attractive force varying as the inverse *fifth power* of the distance (p. 106). The existence of this force depends on the particular atomic hypothesis of pulsating spheres of finite, if small, radii. On that hypothesis a term of the form $\frac{a_1^3 a_{1'}^3}{\gamma_{11'}^4}(a_1 \dot{\phi}_1^2 + a_{1'} \dot{\phi}_{1'}^2)$ as well as the term of form $\frac{\dot{\phi}_1 \dot{\phi}_{1'}}{\gamma_{11'}}$ arises in the force function of the molecules. The former term, however, disappears on the present hypothesis of ether squirts, which reduces the force function solely to terms of the order $\frac{\dot{\phi}_1 \dot{\phi}_{1'}}{\gamma_{11'}}$. Hence we see that on the hypothesis of ether squirts there would be no *molecular force*, when we neglocted the semi-chemical action between molecules, unless the corresponding vibrations of the molecules were in the same phase. Is there any reasoning by which we can prove this result invalid? At first sight, apparently not, especially in our ignorance of the relative magnitude of inter-atomic and inter-molecular distances. In a distended gas the semi-chemical action might be negligible and there would thus be no sensible cohesive force; there is no objection to this.

Such a gas could not be self-incandescent or send forth rays of light without the application of external energy, for this self-incandescence would involve, I think, the vibrations of a great number of molecules being in the *same phase.* On the other hand, it is possible that the means taken to produce the bright line spectrum of such a gas really sets a great number of molecules vibrating in the same phase.* The argument from gaseous bodies giving light, although weighty, is thus not conclusive. If we suppose two free molecules not to be vibrating in the same phase, we are thrown back on a semi-chemical action between the kin-atoms of different molecules as the basis of cohesion. Possibly the effects of altering the chemical constitution of a molecule by increasing the number of like atoms in it, which would affect the cohesion and so the volume of a substance, might tend to throw some light on this point.

(ii). If two free molecules have vibrations in equal phase, or if two kin-atoms have free vibrations of equal phase, and their forced vibrations differ only by terms of the order (atomic radius/atomic distance)^2, then cohesive force varies partly as the inverse square and partly as the inverse fifth power of molecular distance. But so far as these terms are concerned there is no modifying action due to the presence of other molecules in the field (see §§42 and 47 of my first paper).

There is something to be said for the possibility of all equal atoms having started vibrating in the same phase—for what might be termed an instant of 'creation.' The effect of inter-atomic action would be principally felt in a variation of period, and the slight variations in change of phase, if they have any existence, might be of the order (atomic radius/inter-atomic distance)^2 and thus the differences of phase might not amount to a quarter period. It is thus a possibility which we must keep in view in evaluating inter-molecular force, and I shall therefore deal with the above cases under the two headings of 'Equal Phase,' by which I shall mean absolutely equal, or only very small differences, and 'Unequal Phase,' by which I shall connote that the difference of phase may take all possible values.

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* Suppose a number of small tuning-forks at such distances that the feeble air vibrations they produce cannot materially affect each other, although the forks have the same note. They may then all be vibrating in *different* phases. Now a very large tuning-fork, caused to vibrate with this note, might set a whole group of the lesser ones, from which it was about equally distant, vibrating in the *same* phase, and therefore represent the manner in which the spectrum of a gas could arise.

17). Case (i). $[1/m] < [1/a]^3$. (Condition of rare gas?).
Here the expressions for $\dot{\phi}_1$ and $\dot{\phi}_{1'}$ reduce to those of my first paper (p. 105).

Sub-case (a). *Equal Phase.*

The mean values of the expressions $\dot{\phi}_s\dot{\phi}_{s'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$ are now

(r and r′ all values but s and s′.)

$$\dot{\phi}_s\dot{\phi}_{s'} = \tfrac{1}{2}\, C_s C_{s'} n_s^2 + \Sigma \frac{\nu_r^4 n_r^2}{(\nu_r^2 - \nu_s^2)^2}\, \frac{C_r C_{r'}}{2}\, \frac{a_s a_{s'}}{\gamma_{sr}\gamma_{s'r'}},$$

$$\dot{\phi}_q\dot{\phi}_{s'} = -\tfrac{1}{2}\, C_q C_{q'} \frac{\nu_{q'}^2 n_{q'}^2}{\nu_{q'}^2 - \nu_{s'}^2}\, \frac{a_{s'}}{\gamma_{s'q'}} - \tfrac{1}{2}\, C_{s'} C_s \frac{\nu_s^2 n_s^2}{\nu_s^2 - \nu_q^2}\, \frac{a_q}{\gamma_{qs}}$$
$$+ \Sigma \frac{C_r C_{r'}}{2}\, \frac{\nu_r^2 \nu_{r'}^2 n_r^2}{(\nu_r^2 - \nu_q^2)(\nu_{r'}^2 - \nu_{s'}^2)}\, \frac{a_q a_{s'}}{\gamma_{qr}\gamma_{s'r'}}$$

Here we have supposed $n_s = n_{s'}$. Now we have by equation (14),

$$\frac{1}{n_s^2} = \frac{1}{\nu_s^2} - \Sigma \frac{a_s a_r}{\gamma_{sr}^2}\, \frac{1}{\nu_s^2 - \nu_r^2}.$$

Hence in order that $n_{s'} = n_s$, it is as a rule not sufficient that $a_s = a_{s'}$ and $\nu_s = \nu_{s'}$, but we must also have $\gamma_{sr} = \gamma_{s'r'}$, or the structure of the two molecules, at least as far as inter-atomic mean distances are concerned, must be alike. Assuming this to be true, we can rewrite the above values so far as terms of the order $(a/\gamma)^2$, as

(r all values but s.)

$$\dot{\phi}_s\dot{\phi}_{s'} = \tfrac{1}{2}\, C_s C_{s'} \nu_s^2 - \tfrac{1}{2}\, C_s C_{s'} \Sigma \frac{\nu_s^4}{\nu_r^2 - \nu_s^2}\, \frac{a_s a_r}{\gamma_{sr}^2} + \tfrac{1}{2}\Sigma C_r C_{r'} \frac{\nu_r^6}{(\nu_r^2 - \nu_s^2)^2}\left(\frac{a_s}{\gamma_{sr}}\right)^2,$$

$$\dot{\phi}_q\dot{\phi}_{s'} = -\tfrac{1}{2}\, C_q C_{q'} \frac{\nu_q^4}{\nu_q^2 - \nu_s^2}\, \frac{a_s}{\gamma_{sq}} - \tfrac{1}{2}\, C_{s'} C_s \frac{\nu_s^4}{\nu_s^2 - \nu_q^2}\, \frac{a_q}{\gamma_{sq}}$$
$$+ \tfrac{1}{2}\Sigma C_r C_{r'} \frac{\nu_r^6}{(\nu_r^2 - \nu_q^2)(\nu_r^2 - \nu_s^2)}\, \frac{a_q a_s}{\gamma_{qr}\gamma_{sr}}.$$

Now the corresponding terms in the force function of the two molecules will be

$$U = \frac{1}{4\pi\rho}\Sigma \frac{\dot{\phi}_s\dot{\phi}_{s'}}{\gamma_{ss'}} + \frac{1}{4\pi\rho}\Sigma \frac{\dot{\phi}_q\dot{\phi}_{s'}}{\gamma_{qs'}}.$$

Let χ equal the angle between the directions of $\gamma_{ss'}$ and γ_{sq}, then

$$\gamma_{qs'}^2 = \gamma_{ss'}^2 + \gamma_{qs}^2 - 2\gamma_{ss'}\gamma_{qs}\cos\chi,$$

$$\frac{1}{\gamma_{qs'}} = \frac{1}{\gamma_{ss'}}\left\{1 + \left(\frac{\gamma_{qs}}{\gamma_{ss'}}\right)P_1(\cos\chi) + \left(\frac{\gamma_{qs}}{\gamma_{ss'}}\right)^2 P_2(\cos\chi) + \dots\right\},$$

where $P_r(\cos\chi) =$ the rth Lagrangean coefficient.

Similarly,

$$\frac{1}{\gamma_{qq'}} = \frac{1}{\gamma_{q's}}\left\{1 + \left(\frac{\gamma_{s'q'}}{\gamma_{q's}}\right)P_1(\cos\chi') + \left(\frac{\gamma_{s'q'}}{\gamma_{q's}}\right)^2 P_2(\cos\chi') + \dots\right\}$$

$$= \frac{1}{\gamma_{ss'}}\left\{1 + \left(\frac{\gamma_{sq}}{\gamma_{ss'}}\right)(P_1(\cos\chi) + P_1(\cos\chi')) + \text{higher terms}\right\},$$

where $\chi' =$ the angle between $\gamma_{s'q'}$ and $\gamma_{s'q} =$ angle between $\gamma_{s'q'}$ and $\gamma_{s's}$ to a first approximation. Or, if we wish to retain terms of the order $[1/m]^2$, we might write

$$\frac{1}{\gamma_{qq'}} = \frac{1}{\gamma_{gg'}}\left\{1 + \frac{\gamma_{gq}}{\gamma_{gg'}}(\cos\chi_q + \cos\chi_{q'})\right\}$$

$$\frac{1}{\gamma_{qs'}} = \frac{1}{\gamma_{gg'}}\left\{1 + \frac{\gamma_{gq}}{\gamma_{gg'}}\cos\chi_q + \frac{\gamma_{g's'}}{\gamma_{gg'}}\cos\chi_{s'}\right\},$$

where g and g' are any two corresponding points in the two molecules, which may or may not be the positions of two like atoms s and s' (e. g. the centroids of the molecules) and χ_q is the angle between $\gamma_{gg'}$ and γ_{gq}, $\chi_{s'}$ between $\gamma_{g'g}$ and $\gamma_{g's'}$. *Thus we see that when we do not neglect terms of the order* $[1/m]^2$, *the force between two molecules is a function of aspect, as well as central distance.*

If this be true for bodies in a very diffused state, it will be still more appreciable when we deal with solid bodies. It follows that the law of inter-molecular force adopted by Poisson, Navier and Cauchy, namely, that inter-molecular force is a function only of central distance, ceases to be true for ether squirt molecules. This might suggest that the rari-constant theory of elasticity can never be true, owing to the action of 'aspect,' even if we neglect the still more important influence of 'modifying action.' But it must be remembered that the stresses with which we have to deal are the summations of a great number of isolated inter-molecular forces, and thus *unless there is any symmetrical distribution of molecules with regard to aspect*, the mean values of cos χ_q and cos $\chi_{s'}$, as well as the mean values of cos χ_q and cos $\chi_{s'}$, will disappear. The terms of the order $[1/m]^2$ would thus contain no aspect influence; indeed we should have to

go to terms of the order $[1/m]^4$ to find them. Thus, at least in a rarefied gas the aspect influence would very probably be negligible. Like remarks apply in a lesser extent to the case of a solid, so that, as far as aspect influence is concerned, we might more reasonably treat a body of 'confused crystallisation'—e. g. many of the metals of construction—as a rari-constant solid, and accordingly give it only one elastic constant, than give a crystal 15 instead of 21 elastic constants.*

After this digression, I return to the above expression for the force function and write down its value as far as $[1/m]^2$. We find

$$V = \frac{\mu_1}{\gamma_{gg'}} + \frac{\mu_2}{\gamma_{gg'}^2}, \tag{41}$$

where

$$\mu_1 = \frac{1}{8\pi\rho} \sum_s C_s C_{s'} \nu_s^2 \left[1 - \sum_r \frac{\nu_s^2}{\nu_s^2 - \nu_r^2} \frac{a_r}{\gamma_{sr}} \left(2 - \frac{a_s}{\gamma_{sr}} \right) + \sum_{r,q} \frac{\nu_s^4}{(\nu_s^2 - \nu_r^2)(\nu_s^2 - \nu_q^2)} \frac{a_r a_q}{\gamma_{sr}\gamma_{sq}} \right],$$

Σ denoting a summation with regard to all values of s for the first molecule, s' being the corresponding value for the second; Σ, a summation with regard to all values of r except $r = s$; and $\sum_{r,q}$, a summation with regard to *all* values of r and q except s, each term except $r = q$ occurring twice or the whole may be written

$$\left\{ \Sigma \frac{\nu_s^2}{\nu_s^2 - \nu_r^2} \frac{a_r}{\gamma_{sr}} \right\}^2.$$

Thus we may put:

$$\mu_1 = \frac{1}{8\pi\rho} \sum_s C_s C_{s'} \nu_s^2 \left[\left(1 - \sum_r \frac{\nu_s^2}{\nu_s^2 - \nu_r^2} \frac{a_r}{\gamma_{sr}} \right)^2 + \sum_r \frac{\nu_s^2}{\nu_s^2 - \nu_r^2} \frac{a_r a_s}{\gamma_{sr}^2} \right]. \tag{42}$$

Further, we have for μ_2:

$$\mu_2 = \frac{1}{8\pi\rho} \sum_s C_s C_{s'} \Big[\gamma_{gs} (\cos\chi_s + \cos\chi_{s'}) - \sum_r \frac{\nu_s^4}{\nu_s^2 - \nu_r^2} \frac{a_r}{\gamma_{sr}} (\gamma_{gr}\cos\chi_r + \gamma_{g's'}\cos\chi_{s'})$$
$$- \sum_{r'} \frac{\nu_s^4}{\nu_s^2 - \nu_{r'}^2} \frac{a_{r'}}{\gamma_{r's'}} (\gamma_{gs}\cos\chi_s + \gamma_{g'r'}\cos\chi_{r'}) \Big],$$

*I have treated the problems of 'modified action' and 'molecular aspect' at some length in the *History of Elasticity*. See Vol. I, Arts. 921–931, particularly the last article, Art. 1527 (Jellett's views), and Vol. II, Art. 276 (where I have endeavored to show that the Boscovichian atom does not exclude the possibility of multi-constancy, a result confirmed by the 'ether squirt,' which is Boscovichian), and Arts. 304–6 (where I have criticised Saint-Venant's views in the light of 'aspect' and 'modified action'.)

or,

$$\mu_2 = \frac{1}{8\pi\rho} \sum_s C_s C_{s'} \left[\gamma_{gs} (\cos\chi_s + \cos\chi_{s'}) \left(1 - \sum_r \frac{\nu_s^4}{\nu_s^2 - \nu_r^2} \frac{a_r}{\gamma_{rs}}\right) - \sum_{r,r'} \frac{\nu_s^4}{\nu_s^2 - \nu_r^2} \frac{a_r}{\gamma_{sr}} \gamma_{gr} (\cos\chi_r + \cos\chi_{r'}) \right], \qquad (43)$$

where $\sum\limits_s$ denotes a summation with regard to all values of s, $\sum\limits_r$ with regard to all values of r except $r = s$, and $\sum\limits_{r,r'}$ with regard to all values of r (r' being given the corresponding value for the second molecule)—except those corresponding to $r = s$, $r' = s'$.

Equations (41)–(43) lead us to the following results:

The force between two molecules of a substance, which are at such distances that $[1/m]$ may be neglected as compared with $[1/a]^3$ is, on the hypothesis of equal phases—

a). Partly attractive, varying as the inverse square. This follows from the fact that the first term in the square brackets of μ_1 is positive, and this first term is by far the greater.

b). Partly varying as the inverse cube. This latter term in the force depends upon the influence of aspect, and its mean value would probably be zero for an amorphic body. In such a body it might influence the action between two molecules for a time, but it would not be sensible when we deal with the average of large numbers of molecules.

c). There is no modifying action between any third molecule and any other pair.

As we have supposed the molecules at a very great distance as compared with the atomic distances, it may be doubted whether these results would apply to any substance but a rarefied gas, and the first part of the inter-molecular force will then be very slight, and the second probably quite insensible.

If the phases only differ very slightly we shall obtain the same results by replacing $C_s C_{s'}$ by $C_s C_{s'} \cos(\alpha_s - \alpha_{s'})$, $\alpha_s - \alpha_{s'}$ being now a small angle and its cosine positive.

Sub-case (b). *Unequal Phase.*

The action between any pair of molecules will be the same as before, but $C_s C_{s'}$ must be replaced by $C_s C_{s'} \cos(\alpha_s - \alpha_{s'})$, and this latter expression may be either positive or negative. Thus there would be a tendency between some

molecules to cohere but in others to mutual repulsion. Cohesion of the material as a whole would be impossible. Thus this tendency of some molecules to repel each other might be of service in the explanation of diffusive and evaporative phenomena. I do not know that there is any physical reason against its possibility in the cases of gas or liquid.

18). Case (ii). Suppose $[1/m]$ of the same order as $[1/a]^3$. Now the periods n_r, $n_{r'}$, $n_{r''}$ will all be modified *equally* by the influence of kin-atoms. This variation is given by equation (17) for $\delta\mu$. We have

$$\left.\begin{aligned} 1/n_r^2 &= 1/\nu_r^2 - \mu_r - \delta\mu_r \\ &= 1/\nu_r^2 - \sum_s \frac{a_r a_s}{\gamma_{rs}^2} \frac{1}{\nu_r^2 - \nu_s^2} - \left\{ \frac{\mu_r}{\nu_r^4} \left(\frac{a_{r'} a_{r''}}{\gamma_{r'r''}^2} + \frac{a_r a_{r''}}{\gamma_{rr''}^2} + \frac{a_r a_{r'}}{\gamma_{rr'}^2} \right) \right\}^{\frac{1}{3}}, \end{aligned}\right\} \qquad (44)$$

where Σ denotes a summation with regard to all values of s but r.

This expression for $1/n_r^2$ is the same for each of the three molecules.

Returning to equations (39)–(40), we have to calculate the mean values of such types as $\dot{\phi}_s\dot{\phi}_{s'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$. With some reductions I find

$$\left.\begin{aligned} \dot{\phi}_s\dot{\phi}_{s'} = &\frac{n_s^2 C_s C_{s'}}{2} \cos(\alpha_s - \alpha_{s'}) + \frac{n_s^2 \kappa_s' C_s^2}{2} + \frac{n_s^2 \kappa_{s'} C_{s'}^2}{2} + \frac{n_s^2 \kappa_{s''}' C_s C_{s''}}{2} \cos(\alpha_s - \alpha_{s''}) \\ &+ \frac{n_s^2 \kappa_{s''} C_{s'} C_{s''}}{2} \cos(\alpha_{s'} - \alpha_{s''}) + \frac{n_s^2 \kappa_{s''} \kappa_{s''}' C_{s''}^2}{2} + \frac{n_s^2 \kappa_{s'} \kappa_s' C_s C_{s'}}{2} \cos(\alpha_s - \alpha_{s'}) \\ &+ \frac{n_s^2 \kappa_{s'} \kappa_{s''}' C_{s'} C_{s''}}{2} \cos(\alpha_{s'} - \alpha_{s''}) + \frac{n_s^2 \kappa_s' \kappa_{s''} C_s C_{s''}}{2} \cos(\alpha_s - \alpha_{s''}) \\ &+ \sum_{r,r'} \frac{n_r^2 \nu_r^4}{(\nu_r^2 - \nu_s^2)^2} \frac{a_s a_{s'}}{\gamma_{sr} \gamma_{s'r'}} \frac{C_r C_{r'}}{2} \cos(\alpha_r - \alpha_{r'}) \\ &- \sum_r \frac{n_r^2 \nu_r^2}{\nu_r^2 - \nu_s^2} \frac{a_s}{\gamma_{sr}} \lambda_{s'r}' \frac{C_r^2}{2} - \sum_{r'} \frac{n_r^2 \nu_{r'}^2}{\nu_{r'}^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{s'r'}} \lambda_{sr'} \frac{C_{r'}^2}{2} \\ &+ \sum_{r,r'} n_r^2 \lambda_{sr'} \lambda_{s'r}' \frac{C_r C_{r'}}{2} \cos(\alpha_r - \alpha_{r'}) + \sum_{r,r''} n_r^2 \lambda_{sr''} \lambda_{s'r''}' \frac{C_r''}{2} \\ &- \sum_{r,r''} \frac{n_r^2 \nu_r^2}{\nu_r^2 - \nu_s^2} \frac{a_s}{\gamma_{sr}} \lambda_{s'r''}' C_r C_{r''} \cos(\alpha_r - \alpha_{r''}) \\ &- \sum_{r',r''} \frac{n_r^2 \nu_{r'}^2}{\nu_{r'}^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{s'r'}} \lambda_{sr''} \frac{C_{r'} C_{r''}}{2} \cos(\alpha_{r'} - \alpha_{r''}) \\ &+ \sum_{r',r''} n_r^2 \lambda_{sr'} \lambda_{s'r''} \frac{C_{r'} C_{r''}}{2} \cos(\alpha_{r'} - \alpha_{r''}) + \sum_{r,r''} n_r^2 \lambda_{s'r}' \lambda_{sr''} \frac{C_r C_{r''}}{2} \cos(\alpha_r - \alpha_{r''}^2) \end{aligned}\right\} \cdot (45)$$

Here $\sum_r$ is a summation with regard to all the atoms of the first molecule except those which make the subject of the summation infinite. $\sum_{r,r'}$ signifies a

summation for kin-atoms of the first and second molecule with the like restriction. The other symbols have corresponding meanings.

$$
\begin{aligned}
\dot{\phi}_q\dot{\phi}_{s'} = & -\frac{n_s^2\nu_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{qs}}\,\frac{C_{s'}C_s}{2}\cos(\alpha_{s'}-\alpha_s) \\
& -\frac{n_q^2\nu_{q'}^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\frac{C_{q'}C_q}{2}\cos(\alpha_q-\alpha_{q'}) \\
& +\sum_{r,r'}\frac{n_r^2\nu_r^4}{(\nu_r^2-\nu_{s'}^2)(\nu_r^2-\nu_q^2)}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\frac{a_q}{\gamma_{qr}}\,\frac{C_rC_{r'}}{2}\cos(\alpha_r-\alpha_{r'}) \\
& -\frac{\nu_s^2n_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{qs}}\,\varkappa_s'\,\frac{C_s^2}{2}-\frac{\nu_{q'}^2n_q^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{q's'}}\,\varkappa_{q'}\,\frac{C_{q'}^2}{2} \\
& -\frac{n_q^2\nu_{q'}^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{q's'}}\,\varkappa_{q''}\,\frac{C_{q'}C_{q''}}{2}\cos(\alpha_{q'}-\alpha_{q''}) \\
& -\frac{n_s^2\nu_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{qs}}\,\varkappa_{s''}'\,\frac{C_{s''}C_s}{2}\cos(\alpha_s-\alpha_{s''}) \\
& +n_s^2\lambda_{qs'}\frac{C_{s'}^2}{2}+n_q^2\lambda_{s'q}'\frac{C_q^2}{2}+n_s^2\lambda_{qs''}\frac{C_{s''}^2}{2}\varkappa_{s''}'+n_q^2\lambda_{s'q''}'\frac{C_{q''}^2}{2}\varkappa_{q''} \\
& +n_s^2\lambda_{qs''}\frac{C_{s'}C_{s''}}{2}\cos(\alpha_{s'}-\alpha_{s''})+n_q^2\lambda_{s'q''}'\frac{C_qC_{q''}}{2}\cos(\alpha_q-\alpha_{q''}) \\
& +n_s^2\varkappa_s'\lambda_{qs'}\frac{C_sC_{s'}}{2}\cos(\alpha_s-\alpha_{s'})+n_q^2\varkappa_{q'}\lambda_{s'q}'\frac{C_qC_{q'}}{2}\cos(\alpha_q-\alpha_{q'}) \\
& +n_s^2\varkappa_s'\lambda_{qs''}\frac{C_sC_{s''}}{2}\cos(\alpha_s-\alpha_{s''})+n_q^2\varkappa_{q''}\lambda_{s'q}'\frac{C_qC_{q''}}{2}\cos(\alpha_q-\alpha_{q''}) \\
& +n_s^2\varkappa_{s''}'\lambda_{qs'}\frac{C_{s''}C_{s'}}{2}\cos(\alpha_{s''}-\alpha_{s'})+n_q^2\varkappa_{q'}\lambda_{s'q''}'\frac{C_{q'}C_{q''}}{2}\cos(\alpha_{q'}-\alpha_{q''}) \\
& -\sum_{r'}\frac{n_r^2\nu_{r'}^2}{\nu_{r'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\lambda_{qr'}\,\frac{C_{r'}^2}{2}-\sum_r\frac{n_r^2\nu_r^2}{\nu_r^2-\nu_q^2}\,\frac{a_q}{\gamma_{qr}}\,\lambda_{s'r}'\,\frac{C_r^2}{2} \\
& -\sum_{r',r''}\frac{n_r^2\nu_r^2}{\nu_{r'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\lambda_{qr''}\,\frac{C_{r'}C_{r''}}{2}\cos(\alpha_{r'}-\alpha_{r''}) \\
& -\sum_{r,r''}\frac{n_r^2\nu_r^2}{\nu_r^2-\nu_q^2}\,\frac{a_q}{\gamma_{qr}}\,\lambda_{s'r''}'\,\frac{C_rC_{r''}}{2}\cos(\alpha_r-\alpha_{r''}) \\
& +\sum_{r,r'}\lambda_{s'r}'\lambda_{qr'}\frac{C_rC_{r'}}{2}\cos(\alpha_r-\alpha_{r'})+\sum_{r,r''}\lambda_{qr''}\lambda_{s'r}'\frac{C_rC_{r''}}{2}\cos(\alpha_r-\alpha_{r''}) \\
& +\sum_{r',r''}\lambda_{s'r''}'\lambda_{qr'}\frac{C_{r'}C_{r''}}{2}\cos(\alpha_{r'}-\alpha_{r''})+\sum_{r''}\lambda_{s'r''}'\lambda_{qr''}\frac{C_{r''}^2}{2}
\end{aligned}
\tag{46}
$$

The summations are of the same nature as in (45). These formulae appear to give extremely complex results for the types of force function terms due to kin- and non-kin-atoms in different molecules. But this complexity is very

much simplified if we only retain the more important terms.* In order to do this we must determine the order of the $\varkappa$'s and λ's on the supposition that $[1/m] = [1/a]^3$; in this case the types for $\varkappa$ and λ will be those of equations (35) and (36), and $\delta\mu$ is of the order $[1/a]^{\frac{8}{3}}$.

$$\begin{array}{ll} \varkappa_{s'}\,,\ \varkappa_{s''}\,,\ \varkappa'_{s}\,,\ \varkappa'_{s''} & \text{are of the order } [1/a]^{\frac{1}{3}}, \\ \lambda_{sr'},\ \lambda_{sr''},\ \lambda'_{s'r},\ \lambda'_{s'r''} & \quad\text{“}\qquad\text{“}\qquad\text{“}\quad [1/a]^{\frac{4}{3}}. \end{array}$$

Thus, if we only retain terms in the expressions $\dot{\phi}_s\dot{\phi}_{s'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$ so far as the order $[1/a]^2$, this will mean retaining terms up to $[1/a]^5$ in the force function and up to $[1/a]^8$ in the force. We see at once that the terms involving the λ's are all either of the order $[1/a]^{\frac{7}{3}}$ or $[1/a]^{\frac{8}{3}}$ and may be neglected in $\dot{\phi}_s\dot{\phi}_{s'}$. In $\dot{\phi}_q\dot{\phi}_{s'}$ we shall still, however, have to keep the long series of terms with products of $\varkappa$'s and λ's as these are of the order $[1/a]^{\frac{5}{3}}$. Further, since $\delta\mu$ is of the order $[1/a]^{\frac{8}{3}}$, we need only substitute for n_s^2, $\nu_s^2(1+\nu_s^2\mu_s)$, and this in the first term of $\dot{\phi}_s\dot{\phi}_{s'}$ only.

We find

$$\left.\begin{aligned}
\dot{\phi}_s\dot{\phi}_{s'} = {} & \frac{\nu_s^2 C_s C_{s'}}{2}\cos(\alpha_s-\alpha_{s'})\left\{1+\sum_r \frac{a_s a_r}{\gamma_{sr}^2}\,\frac{\nu_s^2}{\nu_s^2-\nu_r^2}\right\} \\
& -\frac{a_{s'}}{\gamma_{ss'}}\,\frac{1}{\delta\mu_s}\,\frac{C_s^2}{2}-\frac{a_s}{\gamma_{ss'}}\,\frac{1}{\delta\mu_{s'}}\,\frac{C_{s'}^2}{2} \\
& -\frac{a_{s'}}{\gamma_{s''s'}}\,\frac{1}{\delta\mu_{s''}}\,\frac{C_s C_{s''}}{2}\cos(\alpha_s-\alpha_{s''})-\frac{a_s}{\gamma_{ss''}}\,\frac{1}{\delta\mu_{s''}}\,\frac{C_{s'}C_{s''}}{2}\cos(\alpha_{s'}-\alpha_{s''}) \\
& +\frac{a_s}{\gamma_{ss''}}\,\frac{a_{s'}}{\gamma_{s's''}}\,\frac{1}{\nu_s^2}\,\frac{1}{(\delta\mu_{s''})^2}\,\frac{C_{s''}^2}{2}+\frac{a_s a_{s'}}{\gamma_{ss'}^2}\,\frac{1}{\nu_s^2}\,\frac{1}{\delta\mu_s\delta\mu_{s'}}\,\frac{C_{s'}C_{s''}}{2}\cos(\alpha_{s'}-\alpha_{s''}) \\
& +\frac{a_s}{\gamma_{ss'}}\,\frac{a_{s'}}{\gamma_{s's''}}\,\frac{1}{\nu_s^2}\,\frac{1}{\delta\mu_{s'}\delta\mu_{s''}}\,\frac{C_{s'}C_{s''}}{2}\cos(\alpha_{s'}-\alpha_{s''}) \\
& +\frac{a_s}{\gamma_{ss''}}\,\frac{a_{s'}}{\gamma_{ss'}}\,\frac{1}{\nu_s^2}\,\frac{1}{\delta\mu_{s''}\delta\mu_s}\,\frac{C_s C_{s''}}{2}\cos(\alpha_s-\alpha_{s''}) \\
& +\sum_{r,r'}\frac{\nu^6}{(\nu_r^2-\nu_s^2)^2}\,\frac{a_s a_{s'}}{\gamma_{sr}\gamma_{s'r'}}\,\frac{C_r C_{r'}}{2}\cos(\alpha_r-\alpha_{r'})
\end{aligned}\right\}. \quad (47)$$

This result shows us that if we were only to retain terms of the order $[1/a]^{\frac{4}{3}}$, the

* The terms neglected may correspond to important physical or chemical phenomena, but at present we must seek only the more general results of the theory. The influence of one molecule on another will in itself be as complex as the action of one planetary system on another supposed in its neighborhood; the complete disturbing action of the whole system on one planet would be not more difficult to determine than that of the action of the two molecules on one of their atoms.

non-kin-atoms would have no influence at all on the force between kin-atoms, for the terms they introduce are only of the order $[1/a]^4$.

Further we find:

$$
\begin{aligned}
\dot{\phi}_q\dot{\phi}_{s'} = & -\frac{\nu_s^4}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{qs}}\frac{C_{s'}C_s}{2}\cos(\alpha_{s'}-\alpha_s) \\
& -\frac{\nu_{q'}^4}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{s'q'}}\frac{C_{q'}C_q}{2}\cos(\alpha_q-\alpha_{q'}) \\
& +\Sigma\frac{\nu_r^6}{(\nu_{r'}^2-\nu_{s'}^2)(\nu_r^2-\nu_q^2)}\frac{a_{s'}}{\gamma_{s'r'}}\frac{a_q}{\gamma_{qr}}\frac{C_rC_{r'}}{2}\cos(\alpha_r-\alpha_{r'}) \\
& +\frac{\nu_s^2}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{qs}}\frac{a_{s'}}{\gamma_{ss'}}\frac{1}{\delta\mu_s}\frac{C_s^2}{2}+\frac{\nu_q^2}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_s}{\gamma_{q's'}}\frac{a_q}{\gamma_{qq'}}\frac{1}{\delta\mu_q}\frac{C_{q'}^2}{2} \\
& +\frac{\nu_{q'}^2}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{q's'}}\frac{a_q}{\gamma_{qq''}}\frac{1}{\delta\mu_{q''}}\frac{C_qC_{q''}}{2}\cos(\alpha_{q'}-\alpha_{q''}) \\
& +\frac{\nu_s^2}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{qs}}\frac{a_{s'}}{\gamma_{s''s'}}\frac{1}{\delta\mu_{s''}}\frac{C_{s''}C_s}{2}\cos(\alpha_s-\alpha_{s''}) \\
& +\frac{\nu_s^2}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{qs}}\frac{a_s}{\gamma_{ss'}}\frac{1}{\delta\mu_{s'}}\frac{C_{s'}^2}{2}+\frac{\nu_q^2}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{q's'}}\frac{a_{q'}}{\gamma_{qq'}}\frac{1}{\delta\mu_q}\frac{C_q^2}{2} \\
& +\frac{\nu_s^2}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{sq}}\frac{a_s}{\gamma_{ss''}}\frac{1}{\delta\mu_{s''}}\frac{C_{s'}C_{s''}}{2}\cos(\alpha_{s'}-\alpha_{s''}) \\
& +\frac{\nu_q^2}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{q's'}}\frac{a_{q'}}{\gamma_{q'q''}}\frac{1}{\delta\mu_{q''}}\frac{C_qC_{q''}}{2}\cos(\alpha_q-\alpha_{q''}) \\
& -\frac{1}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{sq}}\frac{a_s}{\gamma_{ss''}}\frac{a_{s'}}{\gamma_{s's''}}\frac{1}{\delta\mu_{s''}^2}\frac{C_{s''}^2}{2} \\
& -\frac{1}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{q's'}}\frac{a_{q'}}{\gamma_{q'q''}}\frac{a_q}{a_{qq''}}\frac{1}{\delta\mu_{q''}^2}\frac{C_{q''}^2}{2} \\
& -\frac{1}{\nu_{s'}^2-\nu_{q'}^2}\frac{a_q}{\gamma_{qs}}\frac{a_s}{\gamma_{ss'}}\frac{a_{s'}}{\gamma_{ss'}}\frac{1}{\delta\mu_s\delta\mu_{s'}}\frac{C_sC_{s'}}{2}\cos(\alpha_s-\alpha_{s'}) \\
& -\frac{1}{\nu_{s''}^2-\nu_{q'}^2}\frac{a_q}{\gamma_{qs}}\frac{a_s}{\gamma_{ss''}}\frac{a_{s'}}{\gamma_{ss'}}\frac{1}{\delta\mu_s\delta\mu_{s''}}\frac{C_sC_{s''}}{2}\cos(\alpha_s-\alpha_{s''}) \\
& -\frac{1}{\nu_{s'}^2-\nu_{q'}^2}\frac{a_q}{\gamma_{qs}}\frac{a_s}{\gamma_{ss'}}\frac{a_{s'}}{\gamma_{s's''}}\frac{1}{\delta\mu_{s'}\delta\mu_{s''}}\frac{C_{s''}C_{s'}}{2}\cos(\alpha_{s'}-\alpha_{s''}) \\
& -\frac{1}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{s'q'}}\frac{a_{q'}}{\gamma_{qq'}}\frac{a_q}{\gamma_{qq'}}\frac{1}{\delta\mu_q\delta\mu_{q'}}\frac{C_qC_{q'}}{2}\cos(\alpha_q-\alpha_{q'}) \\
& -\frac{1}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{s'q'}}\frac{a_{q'}}{\gamma_{qq'}}\frac{a_q}{\gamma_{qq''}}\frac{1}{\delta\mu_q\delta\mu_{q''}}\frac{C_qC_{q''}}{2}\cos(\alpha_q-\alpha_{q''}) \\
& -\frac{1}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{s'q'}}\frac{a_{q'}}{\gamma_{q'q''}}\frac{a_q}{\gamma_{qq'}}\frac{1}{\delta\mu_{q'}\delta\mu_{q''}}\frac{C_{q'}C_{q''}}{2}\cos(\alpha_{q'}-\alpha_{q''})
\end{aligned}
\qquad (48)
$$

Sub-case (a). *Unequal Phase.*

We suppose that for a great number of molecules the mean difference of phase takes all values from 0 to 2π. Thus although terms like $\cos(\alpha_s - \alpha_{s'})$ may have a finite value for any pair of molecules, yet if we are considering the elastic or cohesive stress across an elementary plane area at a point, such terms will disappear owing to our having to take the mean for a very great number of molecules. We may therefore omit them from the beginning, if we remember that we are not then dealing with any individual pair of molecules, but only with so much of the force between them as will appear after summing for a very great number of molecules. Thus the atoms, or some of them, in one molecule may attract or repel the atoms, or some of them, in a second molecule according as to the magnitude of the difference in phase, but we shall have the following results for $\dot{\phi}_s\dot{\phi}_{s'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$ if we take the mean value for a very great number of molecules of their mean time values as given in (47) and (48)

$$\dot{\phi}_s\dot{\phi}_{s'} = -\frac{a_{s'}}{\gamma_{ss'}}\frac{1}{\delta\mu_s}\frac{C_s^2}{2} - \frac{a_s}{\gamma_{ss'}}\frac{1}{\delta\mu_{s'}}\frac{C_{s'}^2}{2} + \frac{a_s a_{s'}}{\gamma_{ss''}\gamma_{s's''}}\frac{1}{\nu_s^2}\frac{1}{(\delta\mu_{s''})^2}\frac{C_{s''}^2}{2}\,. \tag{49}$$

$$\left.\begin{aligned}\dot{\phi}_q\dot{\phi}_{s'} = {} & \frac{\nu_s^2}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{qs}}\frac{a_{s'}}{\gamma_{ss'}}\frac{1}{\delta\mu_s}\frac{C_s^2}{2} + \frac{\nu_q^2}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{q's'}}\frac{a_q}{\gamma_{qq'}}\frac{1}{\delta\mu_{q'}}\frac{C_{q'}^2}{2} \\ & + \frac{\nu_s^2}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{qs}}\frac{a_s}{\gamma_{ss'}}\frac{1}{\delta\mu_{s'}}\frac{C_{s'}^2}{2} + \frac{\nu_q^2}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{q's'}}\frac{a_{q'}}{\gamma_{qq'}}\frac{1}{\delta\mu_q}\frac{C_q^2}{2} \\ & - \frac{1}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{qs}}\frac{a_s}{\gamma_{ss''}}\frac{a_{s'}}{\gamma_{s's''}}\frac{1}{(\delta\mu_{s''})^2}\frac{C_{s''}^2}{2} \\ & - \frac{1}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{q's'}}\frac{a_{q'}}{\gamma_{q'q''}}\frac{a_q}{\gamma_{qq''}}\frac{1}{(\delta\mu_{q''})^2}\frac{C_{q''}^2}{2}\end{aligned}\right\}. \tag{50}$$

The terms in the expression for $\dot{\phi}_s\dot{\phi}_{s'}$ are of the order $[1/a]^{\frac{1}{3}}$ and $[1/a]^{\frac{2}{3}}$, while those in that for $\dot{\phi}_q\dot{\phi}_{s'}$ $[1/a]^{\frac{4}{3}}$ and $[1/a]^{\frac{5}{3}}$. The former gives rise to the more important terms in the force function. Thus the most important terms of inter-molecular force are given by the force function

$$\left.\begin{aligned}U = {} & -\sum_{s,\,s',\,s''}\frac{1}{8\pi\rho}\frac{a_s}{\gamma_{ss'}^2}(C_s^2 + C_{s'}^2)\frac{1}{\left\{\dfrac{\mu_s}{\nu_s^4}\left(\dfrac{a_s^2}{\gamma_{s's''}^2} + \dfrac{a_s^2}{\gamma_{ss''}^2} + \dfrac{a_s^2}{\gamma_{ss'}^2}\right)\right\}^{\frac{1}{3}}} \\ & + \sum_{s,\,s',\,s''}\frac{1}{8\pi\rho}\frac{a_s^2}{\gamma_{ss'}\gamma_{ss''}\gamma_{s's''}}C_{s''}^2\frac{1}{\left\{\dfrac{\mu_s}{\nu_s^4}\left(\dfrac{a_s^2}{\gamma_{s's''}^2} + \dfrac{a_s^2}{\gamma_{ss''}^2} + \dfrac{a_s^2}{\gamma_{ss'}^2}\right)\right\}^{\frac{2}{3}}}\end{aligned}\right\}, \tag{51}$$

To free this result from the great variety of inter-atomic distances we ought to replace $1/\gamma_{ss'}$, etc., by expressions in terms of $1/\gamma_{gg'}$, etc., as in Art. 17, but we may replace $1/\gamma_{ss'}$ straight away by $1/\gamma_{gg'}$ if we neglect terms of the order $[1/a]^{\frac{19}{3}}$, retaining those of the order $[1/a]^{\frac{14}{3}}$. Thus we can rewrite the above expression:

$$\left.\begin{aligned} U = &-\frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}^2}\,\frac{1}{\left\{\dfrac{1}{\gamma_{g'g''}^2}+\dfrac{1}{\gamma_{gg''}^2}+\dfrac{1}{\gamma_{gg'}^2}\right\}^{\frac{1}{3}}}\sum_{s,s'}\left(C_s^2+C_{s'}^2\right)\left(\frac{\nu_s^4 a_s}{\mu_s}\right)^{\frac{1}{3}} \\ &+\frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}\gamma_{gg''}\gamma_{g'g''}}\,\frac{1}{\left\{\dfrac{1}{\gamma_{g'g''}^2}+\dfrac{1}{\gamma_{gg''}^2}+\dfrac{1}{\gamma_{gg'}^2}\right\}^{\frac{2}{3}}}\sum_{s,s''}\left\{\frac{\nu_s^4 a_s}{\mu_s}\right\}^{\frac{2}{3}} C_{s''}^2 \end{aligned}\right\} \quad (52)$$

I will now note the conclusions we can draw from this approximation to the value of U.

a). 'Aspect' does not influence inter-molecular action, at least in the principal terms. Indeed, although we have indicated that it would occur in terms of the order $[1/a]^{\frac{19}{3}}$, yet even here in the case of an amorphic body it would disappear for the mean of a great number of molecules.

b). The modifying effect of the presence of a third molecule on the law of inter-molecular force between two others is not merely a feature of the second or higher approximations, but is fundamental to the first approximation; i. e. so long as the third molecule is at distances from the first two of the same order as their distance, modifying action cannot be neglected.

c). The first term in the force function is of the order $[1/a]^{\frac{10}{3}}$ and *negative*, hence the corresponding force is repulsive; the second term is of the order $[1/a]^{\frac{11}{3}}$, and so it might thus appear that two molecules must always repel each other. But it must be remembered that we have only taken the second term for a *single modifying molecule*, and that for the cohesion the second term would have to be summed over the whole sphere of inter-molecular influence. This summation would cause the term to rise in order, and we thus see that, *if on the hypothesis of unequal phases the force between two molecules be attractive, then this attraction is solely due to the truth of the hypothesis of modified action;* i. e. cohesion depends on the truth of this hypothesis.

d). While we thus see that the chief terms in U depend on the action of kin-atoms, we have also noted that kin-atoms tend to produce a repulsive inter-molecular action, and unless their influence is counteracted by the modifying

action of other molecules, we have a repulsive force between kin-atoms. If we consider the terms (which are of a higher order) in the force between non-kin-atoms and suppose the molecules absolutely alike so that $a_s = a_{s'}$, $C_s = C_{s'}$, etc., we find

$$\frac{\dot{\Phi}_q \dot{\Phi}_s}{\gamma_{qs'}} = \frac{a_q a_s}{\gamma_{qs}} \frac{1}{\gamma^2_{gg'}} \frac{1}{\left\{\frac{1}{\gamma^2_{g'g''}} + \frac{1}{\gamma^2_{gg''}} + \frac{1}{\gamma^2_{gg'}}\right\}^{\frac{1}{3}}} \frac{C_s^2\left(\frac{\nu_s^{10}}{\mu_s a_s^2}\right)^{\frac{1}{3}} - C_q^2\left(\frac{\nu_q^{10}}{\mu_q a_q^2}\right)^{\frac{1}{3}}}{\nu_s^2 - \nu_q^2}$$

to a first approximation.

Hence the force between non-kin atoms of different molecules will be attractive if the atom whose free vibration has the shorter period ($\nu_s > \nu_q$) be of such amplitude in vibrational flow that

$$C_s^2\left(\frac{\nu_s^{10}}{\mu_s a_s^2}\right)^{\frac{1}{3}} > C_q^2\left(\frac{\nu_q^{10}}{\mu_q a_q^2}\right)^{\frac{1}{3}}.$$

Now C_s and C_q will probably depend to a very great extent on the period of any energy which may be in a state of transmission through the ether in the neighborhood of the atom. For example, light or heat energy of a period nearly equal to $2\pi/n_s$ would tend greatly to affect C_s but not C_q, and energy of a period nearly equal to $2\pi/n_q$ would affect C_q and not C_s, thus we should expect the inter-molecular action of non-kin atoms to be even capable of changing its sign owing to the nature of the optic, thermal or electric field in which the molecules are placed. Thus these non-kin atom terms in the force function would produce a secondary influence on the general law of cohesion due to the conditions of the optic, thermal or electric field. Of course these conditions would also affect C_s, $C_{s'}$ and $C_{s''}$, but examining the value of U in (51), we see that to decrease or increase these quantities would, speaking very generally, be rather quantitative; i. e. there would not be a distinctly opposite effect on U for a field in which energy of period $2\pi/n_s$ and a field in which energy of period $2\pi/n_q$ were respectively prominent. The *qualitative* effect therefore of energy of different periods would, if it is in any case sensible, have to be attributed to the action of non-kin atoms and the terms in U arising from (50).

e). Since U consists of terms of opposite sign, and we have indicated that owing to the summation which we must make for a great number of modifying molecules, the second term may rise to equal order and importance with the first (see (c)), it follows that for a certain value of $\gamma_{gg'}$ $dU/d\gamma_{gg'}$ is zero, or *the inter-molecular force changes sign.* This gives, then, a mean inter-molecular distance at which for definite values of the C's, i. e. for definite conditions of the thermal, optic and electric fields, the molecules exert no force on each other. If we bring

the molecules closer together they will repel; if we separate them, they will attract. This is the ordinary phenomena of elastic stress, and this change in the sign of *inter-molecular* force seems an important addition to the discussion of cohesion as given in my first paper. It arises from the terms due to kin-atoms and the modifying action, which were neglected in that paper.

f). The presence of a greater or less number of kin-atoms will obviously greatly influence the law of cohesion, although whether they will tend to increase or decrease the mean inter-molecular distance must depend largely on the nature of the second term of U in (51) when we sum it for *all* the modifying molecules. But the mean inter-molecular distance will settle the volume. Thus we should expect remarkable divergences in the volumetric analysis of quantities like CH_4O, C_2H_6O, C_3H_8O or Cu_2O, Cu_2O_2, etc.; and again, noteworthy variations in the elasticity and cohesion of steel as we introduce more and more carbon.

It seems not improbable that some physical measure of the effect of kin-atoms on cohesion might be obtained from the cohesion of plates of different substances when pressed together, the molecules of the two substances having no kin-atoms, or one or more kin-atoms in their structure.

Sub-case (b). *Unequal Phase.*

20). I now pass to the consideration of the value of the terms in the force function when the phases of kin-atoms in different molecules are either equal or differ by less than $\pi/2$. The physical probability seems that if they do not take all possible values they will only differ slightly (see §47 of my first paper), so that we have only to replace the cosines in (47) and (48) by a quantity ε very nearly equal to unity in order to obtain in this sub-case the mean values of $\dot{\phi}_s\dot{\phi}_{s'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$.

If we retain terms up to $[1/a]^2$ we can omit none of those given in (47) for $\dot{\phi}_s\dot{\phi}_{s'}$; if we retain only terms of order $[1/a]^{\frac{2}{3}}$ we may omit the second term in the bracket of first line and the last term; if we retain only terms of the order $[1/a]^{\frac{1}{3}}$ we preserve only the terms in the first to the third lines, omitting second term of the first. Similarly for $\dot{\phi}_q\dot{\phi}_{s'}$ for

approximation up to $[1/a]^2$ we retain all terms.
" " $[1/a]^{\frac{5}{3}}$ " " but those in the third line.
" " $[1/a]^{\frac{4}{3}}$ " 1st, 2d, 4th, 9th lines only.
" " $[1/a]^{\frac{2}{3}}$ " 1st and 2d lines only.
" " $[1/a]^{\frac{1}{3}}$ we need consider no terms in $\dot{\phi}_q\dot{\phi}_{s'}$.

As we only wish to obtain a general idea of inter-molecular force, let us adopt the approximation to the order $[1/a]^{\frac{1}{3}}$. So far as the terms then dealt with are concerned, *aspect* will not enter into our value of the force function U, as it would involve terms of order $[1/a]^5$, whereas we are only retaining those of order $[1/a]^{\frac{10}{3}}$. We find

$$U = \left. \begin{aligned} & \frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}}\,\sum_{s,\,s'} C_s C_{s'} \varepsilon_{ss'} \\ & - \frac{1}{8\pi\rho}\,\frac{1}{\gamma^2_{gg'}}\,\sum_{s,\,s's''} \left\{ \frac{a_{s'} C_s^2}{\delta\mu_s} + \frac{a_s C_{s'}^2}{\delta\mu_s} \right\} \\ & - \frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}}\,\sum_{s,\,s's''} \left\{ \frac{a_{s'} C_s C_{s''} \varepsilon_{ss''}}{\gamma_{s''s'}\,\delta\mu_{s''}} + \frac{a_s C_{s'} C_{s''} \varepsilon_{s's''}}{\gamma_{ss''}\,\delta\mu_{s''}} \right\} \end{aligned} \right\} \quad (53)$$

We may now deduce some general results from this expression:

a). Neglecting modifying action, the inter-molecular force varies as the inverse square* and is attractive. Thus for this case the force of cohesion could never become negative. Turning to the second and third terms, however, these give *repulsive forces.* The order of the part of the force due to the first term will be $[1/a]^6$, while that due to the second and third of the order $[1/a]^{\frac{19}{3}}$. Thus the difference in order is not very great, and if the modifying action as expressed in the third term be summed for a great number of molecules, this term may rise in order and possibly become so great that the force changes sign and becomes repulsive. The possibility of repulsive inter-molecular force would thus again depend on the law of modified action.

b). A distinction must be drawn between (a) of this article and (c) of the preceding article. For unequal phases the term of least order in the force is repulsive and of order $[1/a]^{\frac{19}{3}}$. To account for a cohesive force and the necessary change in sign, we must suppose the attractive terms in the force of order $[1/a]^{\frac{20}{3}}$ to become all-important on summation for a great number of molecules. For equal phases the term of least order in the force is attractive and of order $[1/a]^6$, and the necessary change of sign to explain cohesion is obtained by supposing terms of the order $[1/a]^{\frac{19}{3}}$ to rise into importance on summation. The latter seems a more reasonable hypothesis, as it would allow of the union of two solitary molecules, which the former would not. The former indeed indicates that

* It must not be supposed that because this force varies as the 'inverse square', that cohesion is thus only a part of gravitating force, and so our theory be liable to the destructive criticism of Belli and others. That criticism is based, not on the power of the distance, but on the magnitude of the constant of variation being the same as in the case of gravitation.

only a certain number of molecules would be able to cohere at all unless they were chosen with particular differences of phase. So far as I know, there are no physical facts which would point to any such selective action on the part of molecules when uniting in small groups to form a solid. At the same time the phenomena of smell tell us that bodies must be continually throwing off and therefore *repelling* their external molecules. This must either mean a sufficient difference in phase to produce a repulsive force between molecules, or else that the modifying action on *superficial* molecules is not always sufficiently great to overcome the repulsive influence produced by kin-atoms. Thus we should have to look upon smell on the hypothesis of nearly equal phase as a process by which individual eccentric or 'unsympathetic' molecules—i. e. those whose atoms had a wide divergence of phase from that of the mean of their kin-atoms in the system—were gradually brought to the surface and ejected; but on the hypothesis of unequal phase, smell would be due to the failure of sufficient "modifying action" at the surface.

c). Remarks (a) and (b) of the previous article apply in this case also, while we may treat the law of force between non-kindred atoms by examining the first two terms of (48). For exactly like molecules we have, to a first approximation, the term in the force function for the qth and s'^{th} atoms,

$$-\frac{1}{\gamma_{gg'}}\,\frac{a_q a_s}{\gamma_{sq}}\,\frac{1}{8\pi\rho}\,\frac{C_s^2\nu_s^4/a_s - C_q^2\nu_q^4/a_q}{\nu_s^2-\nu_q^2},$$

or there is an attractive force between the qth and s'^{th} atoms,

$$=\frac{1}{\gamma_{gg'}^2}\,\frac{\pi}{2\rho}\,\frac{1}{\nu_s^2\nu_q^2}\,\frac{a_q a_s}{\gamma_{sq}}\,\frac{\dfrac{C_s^2\nu_s^4}{a_s}-\dfrac{C_q^2\nu_q^4}{a_q}}{\dfrac{4\pi^2}{\nu_s^2}-\dfrac{4\pi^2}{\nu_q^2}}. \qquad (54)$$

But this force—

$$=\tfrac{1}{2}\left(\frac{\gamma_{sq}}{\gamma_{gg'}}\right)^2\begin{Bmatrix}\text{atomic force between } q^{\text{th}} \text{ and } s^{\text{th}}\\ \text{atoms of one molecule.}\end{Bmatrix} \qquad (55)$$

This follows by (17), if we only retain the lowest powers of $F_{ss'}$. Thus the intermolecular force between the qth and s'^{th} atoms is at once expressed in terms of the atomic force between q^{th} and s^{th} atoms. It will obviously be attractive or repulsive according to the sign of the "chemical affinity" or the last factor on right of (54). *Thus, ceteris paribus, the more difficult its molecule is to disassociate, the greater the cohesion of the substance.*

21). Let us now pass to the third case when inter-molecular distance bears to inter-atomic the ratio of order given by $[1/m] = [1/a]^2$.

In this case $\delta\mu_s$, $\delta\mu_{s'}$, $\delta\mu_{s''}$ are different, being the roots of the cubic equation (26), which we may write

$$\left.\begin{array}{r}\delta\mu^3 - \dfrac{\delta\mu}{\nu_s^4}\left(\dfrac{a_{s'}a_{s''}}{\gamma_{s's''}^2} + \dfrac{a_s a_{s''}}{\gamma_{ss''}^2} + \dfrac{a_s a_{s'}}{\gamma_{ss'}^2}\right) + \dfrac{2}{\nu_s^6}\,\dfrac{a_s a_{s'} a_{s''}}{\gamma_{s's''}\gamma_{ss''}\gamma_{s's}} \\ - \dfrac{\mu_s}{\nu_s^4}\left(\dfrac{a_{s'}a_{s''}}{\gamma_{s's}^2} + \dfrac{a_s a_{s''}}{\gamma_{ss''}^2} + \dfrac{a_s a_{s'}}{\gamma_{ss'}^2}\right) = 0\end{array}\right\}. \quad (56)$$

Thus n_s, $n_{s'}$ and $n_{s''}$ will differ, and thus the mean value of

$$\sin(n_s t + \alpha_s)\,\sin(n_{s'}t + \alpha_{s'})$$

is zero. From equations (39) to (41) we find

$$\begin{aligned}\dot\phi_s\dot\phi_{s'} = {} & \frac{n_s^2\varkappa_s' C_s^2}{2} + \frac{n_{s'}^2\varkappa_{s'} C_{s'}^2}{2} + \frac{n_{s''}^2\varkappa_{s''}\varkappa_{s''}' C_{s''}^2}{2} \\ & - \sum_r \frac{\nu_r^2 n_r^2}{\nu_r^2 - \nu_s^2}\,\frac{a_s}{\gamma_{sr}}\,\lambda_{s'r}'\,\frac{C_r^2}{2} \\ & - \sum_{r'} \frac{\nu_{r'}^2 n_{r'}^2}{\nu_{r'}^2 - \nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\lambda_{sr'}\,\frac{C_{r'}^2}{2} \\ & + \sum_{r''} n_{r''}^2\lambda_{sr''}\lambda_{s'r''}'\,\frac{C_{r''}^2}{2},\end{aligned}$$

$$\begin{aligned}\dot\phi_q\dot\phi_{s'} = {} & -\frac{n_s^2\nu_s^2}{\nu_s^2 - \nu_q^2}\,\frac{a_q}{\gamma_{sq}}\,\varkappa_s'\,\frac{C_s^2}{2} - \frac{n_{q'}^2\nu_{q'}^2}{\nu_{q'}^2 - \nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\varkappa_{q'}\,\frac{C_{q'}^2}{2} \\ & + n_{s'}^2\lambda_{qs'}\frac{C_{s'}^2}{2} + n_q^2\lambda_{s'q}'\frac{C_q^2}{2} + n_{s''}^2\varkappa_{s''}'\lambda_{qs''}\frac{C_{s''}^2}{2} + n_{q''}^2\varkappa_{q''}\lambda_{s'q''}'\frac{C_{s''}^2}{2} \\ & - \sum_{r'} \frac{\nu_{r'}^2 n_r^2}{\nu_{r'}^2 - \nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\lambda_{qr'}\,\frac{C_{r'}^2}{2} - \sum_r \frac{\nu_r^2 n_r^2}{\nu_r^2 - \nu_q^2}\,\frac{a_q}{\gamma_{qr}}\,\lambda_{s'r}'\,\frac{C_r^2}{2} \\ & + \sum_{r''} n_{r''}^2\lambda_{s'r''}'\lambda_{qr''}\,\frac{C_{r''}^2}{2}.\end{aligned}$$

In all cases the summations are taken so as to exclude infinite terms. Substituting the value of the λ's from p. 51, we may rewrite these expressions,

$$\left.\begin{aligned}\dot\phi_s\dot\phi_{s'} = {} & \frac{n_s^2\varkappa_s' C_s^2}{2} + \frac{n_{s'}^2\varkappa_{s'} C_{s'}^2}{2} + \frac{n_{s''}^2\varkappa_{s''}\varkappa_{s''}' C_{s''}^2}{2} \\ & + \sum_{r,r'} \frac{n_r^2\nu_r^4}{(\nu_r^2 - \nu_s^2)^2}\,\frac{a_s}{\gamma_{sr}}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\varkappa_r'\,\frac{C_r^2}{2} \\ & + \sum_{r',r} \frac{n_{r'}^2\nu_{r'}^4}{(\nu_r^2 - \nu_{s'}^2)^2}\,\frac{a_s}{\gamma_{sr}}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\varkappa_{r'}\,\frac{C_{r'}^2}{2} \\ & - \sum_{r,r'r''} \frac{n_{r''}^2\nu_r^4}{(\nu_{r''}^2 - \nu_s^2)(\nu_{r''}^2 - \nu_{s'}^2)}\,\frac{a_s}{\gamma_{rs}}\,\frac{a_{s'}}{\gamma_{r's'}}\,\varkappa_{r''}\varkappa_{r''}''\,\frac{C_{r''}^2}{2}\end{aligned}\right\}. \quad (57)$$

$$\left.\begin{aligned}\dot{\phi}_q\dot{\phi}_{s'} = &- \frac{n_s^2\nu_s^2}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{sq}}\varkappa_s'\frac{C_s^2}{2} - \frac{n_{q'}^2\nu_{q'}^2}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{s'q'}}\varkappa_{q'}\frac{C_{q'}^2}{2}\\ &- \frac{n_{s'}^2\nu_{s'}^2}{\nu_{s'}^2-\nu_{q'}^2}\frac{a_q}{\gamma_{sq}}\varkappa_{s'}\frac{C_{s'}^2}{2} - \frac{n_q^2\nu_q^2}{\nu_q^2-\nu_s^2}\frac{a_{s'}}{\gamma_{s'q'}}\varkappa_q'\frac{C_q^2}{2}\\ &- \left\{\frac{n_{s''}^2\nu_{s''}^2}{\nu_{s''}^2-\nu_{q''}^2}\frac{a_q}{\gamma_{sq}}\varkappa_{s''}'\varkappa_{s''} + \frac{n_{q''}^2\nu_{q''}^2}{\nu_{q''}^2-\nu_{s''}^2}\frac{a_{s'}}{\gamma_{s'q'}}\varkappa_{q''}'\varkappa_{q''}\right\}\frac{C_{s''}^2}{2}\\ &+ \sum_{r,r'}\frac{\nu_r^4 n_r^2}{(\nu_r^2-\nu_s^2)(\nu_r^2-\nu_q^2)}\frac{a_q}{\gamma_{qr}}\frac{a_{s'}}{\gamma_{s'r'}}\varkappa_{r'}\frac{C_{r'}^2}{2}\\ &+ \sum_{r',r}\frac{\nu_r^4 n_r^2}{(\nu_r^2-\nu_q^2)(\nu_r^2-\nu_s^2)}\frac{a_q}{\gamma_{rs}}\frac{a_{s'}}{\gamma_{r's'}}\varkappa_r'\frac{C_r^2}{2}\\ &+ \sum_{r'',r'}\frac{n_{r''}^2\nu_{r''}^4}{(\nu_{r''}^2-\nu_{s'}^2)(\nu_{r''}^2-\nu_q^2)}\frac{a_s}{\gamma_{qs}}\frac{a_{s'}}{\gamma_{r's'}}\varkappa_{r''}'\varkappa_{r''}\frac{C_{r''}^2}{2}\end{aligned}\right\} \quad (58)$$

Now the $\varkappa$'s are of zero order in $[1/a]$. Hence, if we want to find the value of $\dot{\phi}_s\dot{\phi}_{s'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$ to the order $[1/a]^2$, we must substitute for n_s^2,

$$n_s^2 = \nu_s^2\{1 + \nu_s^2(\mu_s + \delta\mu_s)\}$$

in the three first terms of $\dot{\phi}_s\dot{\phi}_{s'}$. Further, in the same terms, when they occur in the force function, we must substitute the value of $\gamma_{ss'}$ given in terms of $\gamma_{gg'}$ on p. 58, thus introducing the *aspect influence.* Owing to the same arguments as I have before used, these terms in the aspect influence containing first powers of cosines (see p. 58) will vanish for an *amorphic* body, when we are considering the law of cohesive force and not the force between any given individual pair of molecules. Thus the aspect influence would not be sensible below terms of the order $[1/a]^6$, which it must, however, be remarked is the order of terms which the second, third and fourth lines of $\dot{\phi}_s\dot{\phi}_{s'}$ contribute to the force.

As we cannot express analytically the solution of the cubic (56), we cannot substitute the value of n_s and $\varkappa_s'$, etc., in terms of $\gamma_{s''s'}$, $\gamma_{s''s}$ and $\gamma_{ss'}$; thus it is impossible to reduce the value of $\dot{\phi}_s\dot{\phi}_{s'}$ to terms in $\gamma_{g'g''}$, $\gamma_{gg''}$ and $\gamma_{g'g}$, and so show the exact nature of the aspect influence. But we may note that the aspect influence when $[1/m]$ is of the order $[1/a]^2$ plays a much larger part in the inter-molecular force than previously. Hence if the nearness of the molecules of a solid is of this order we should expect 'aspect' as well as 'modified action' to contribute towards the probability of multi-constancy.

The whole of the reasoning for this case is independent of the equality or

inequality of the phases. Thus, so far as the hypotheses of 'aspect' and 'modified action' are concerned, we might conclude from the present theory that—

(i). Neither has sensible influence if $[1/m]$ is not comparable with $[1/a]^3$.
(ii). The latter only has " " is comparable with $[1/a]^3$.
(iii). Both have " " " " $[1/a]^2$.

In the third case, however, only the latter action influences the terms of the very first importance in inter-molecular force.

Returning now to the formulae (57) and (58), suppose we retain only the terms to order $[1/a]^3$ in the force function, neglecting also those aspect influence terms which, although of this order, would in the case of an amorphic solid disappear for the mean of a great number of molecules. We find

$$\left.\begin{aligned} U = & \frac{1}{4\pi\rho}\Sigma\frac{\dot{\Phi}_s\dot{\Phi}_{s'}}{\gamma_{ss'}} + \frac{1}{4\pi\rho}\Sigma\frac{\dot{\Phi}_q\dot{\Phi}_{s'}}{\gamma_{qs'}} + \frac{1}{4\pi\rho}\Sigma\frac{\dot{\Phi}_s\dot{\Phi}_{q'}}{\gamma_{sq'}} \\ = & \frac{1}{8\pi\rho}\frac{1}{\gamma_{gg'}}\sum_{s,\,s's''}\{\nu_s^2\varkappa_s' C_s^2 + \nu_{s'}^2\varkappa_{s'}C_{s'}^2 + \nu_{s''}^2\varkappa_{s''}\varkappa_{s''}' C_{s''}^2\} \\ & - \frac{1}{8\pi\rho}\frac{1}{\gamma_{gg'}}\sum_s\sum_q\frac{\nu_s^4}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{sq}}(\varkappa_s' C_s^2 + \varkappa_{s'}C_{s'}^2) \\ & - \frac{1}{8\pi\rho}\frac{1}{\gamma_{gg'}}\sum_s\sum_q\frac{\nu_q^4}{\nu_q^2-\nu_s^2}\frac{a_{s'}}{\gamma_{s'q'}}(\varkappa_{q'}C_{q'}^2 + \varkappa_q' C_q^2) \\ & - \frac{1}{8\pi\rho}\frac{1}{\gamma_{gg'}}\sum_s\sum_q\left(\frac{\nu_s^4}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{sq}}\varkappa_{s''}'\varkappa_{s''} + \frac{\nu_q^4}{\nu_q^2-\nu_s^2}\frac{a_{s'}}{\gamma_{s'q'}}\varkappa_{q''}'\varkappa_{q''}\right)C_{s''}^2 \end{aligned}\right\} \quad (59)$$

Here the first sum is to take every possible value of s with the corresponding values of s' and s'', while the last three double sums are with regard to all values of s and q, except $s = q$.

We cannot express the inter-molecular force as some simple inverse power of inter-molecular distances, as the $\varkappa$'s are complicated functions of those distances involving the modifying action. We are still able, however, to deduce some general results from the above value of U.

a). The force between two molecules (when $[1/m]$ is of the order $[1/a]^2$, depends, so far as its chief terms are concerned, on the influence of kin-atoms) either in that which they exert directly on each other or on that indirect influence which they have in forcing pulsations of their own mutually modified periods upon their colleagues in their respective groups.

b). If we examine the values of the $\varkappa$'s given Art. 16, we see that if $[1/m]$ be not of order $[1/a]^2$, they are negative, as the second terms in both numerator

and denominator are small as compared with the first. If the second term in either grows in importance, $\varkappa$ will still remain negative so long as our approximation to its value remains true, or otherwise $\varkappa$ would pass through either zero or infinity, neither of which is admissible. Thus the first two terms of the first line of U will remain negative. Hence the force will be repulsive unless the third term summed for every modifying molecule rises to a greater value than the first two. Now $\varkappa$ will I think always be necessarily less than unity, hence the third term of the order $\varkappa^2$ is less than the first two in order of magnitude. It may or may not, however, rise owing to the summation for all modifying molecules. The possibility, I should prefer to say probability, of the inter-molecular force becoming negative for $[1/m] = [1/a]^2$ is, however, sufficiently clear.

22). If the above discussion on cohesion has seemed to a certain extent inclusive, this is only partially due to the complexity of the formulae, which, under any theory of cohesion, are bound to be complex if they are to explain the enormous variety of physical phenomena which we may include under that head. The real difficulty of the discussion lies rather in our ignorance as to whether—

α). Like atoms in different molecules have the same or nearly the same phase or not.

β). Molecular distance is comparable with atomic, and if so, what is the exact degree of the comparison.

The discussion, however, seems to have brought out a good deal that remained obscure in the treatment of cohesion in my first paper. It has, I think, shown that the ether squirt theory—

α). Leads to multi-constant equations of elasticity as a result rather of the truth of the 'hypothesis of modified action' than of that of 'aspect.'

β). Suggests that it is to the mutual influence of kin-atoms in different molecules that we must look for explanation of many of the facts of cohesion as well as for the origin of bands and other phenomena of spectrum analysis.

γ). Accounts for a change in sign as well as for a total change in character of the law of inter-molecular force as we vary the mean inter-molecular distance from something incomparably greater than inter-atomic distance down to distances of the order $[1/a]^2$.

δ). Enables us finally to lay aside for heavy matter all notion of 'force or

action at a distance.' All such force is only apparent, arising from the terms in the kinetic energy of the whole system due to the insensible motion of the ether.

23). It must be remembered that the conclusions of this paper do not replace but supplement the results obtained in the three previous papers. The laws of chemical action and the theory of the spectrum given in the first paper, the theory of optic and magnetic phenomena developed in the second paper, and lastly, the equations of generalized elasticity as discussed in the third paper, all flow as readily from the ether squirt as from the pulsating spherical atom. The former atom is, therefore, even better suited than the latter to be used as a model dynamical system for deducing the differential equations which express the laws of physical phenomena. It is, some may think, unlikely that the molecule is really a group of ether squirts, but the molecule is a dynamical system, and any model of a molecule which does not contradict obvious physical facts, but goes a long way to explain those facts, cannot but be suggestive as to the nature of the laws governing real molecular systems. The modifying action of kin-atoms in like molecules seems a suggestion of this kind sufficiently valuable to make it worth while to study closely the ether squirt hypothesis.

The application of this hypothesis to the generalized equations of elasticity and to the laws connecting pressure and volume in a gas must be deferred for the present, considering the great length to which this paper has already run.